The Annals of Probability
2012, Vol. 40, No. 4, 1436–1482
DOI: 10.1214/11-AOP655

QUENCHED ASYMPTOTICS FOR BROWNIAN MOTION OF RENORMALIZED POISSON POTENTIAL AND FOR THE RELATED PARABOLIC ANDERSON MODELS

BY XIA CHEN[1]

University of Tennessee

Let B_s be a d-dimensional Brownian motion and $\omega(dx)$ be an independent Poisson field on $\mathbb{R}^d$. The almost sure asymptotics for the logarithmic moment generating function

$$\log \mathbb{E}_0 \exp\left\{\pm\theta \int_0^t \overline{V}(B_s)\,ds\right\} \qquad (t \to \infty)$$

are investigated in connection with the renormalized Poisson potential of the form

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}[\omega(dy) - dy], \qquad x \in \mathbb{R}^d.$$

The investigation is motivated by some practical problems arising from the models of Brownian motion in random media and from the parabolic Anderson models.

1. Introduction. Consider a particle doing a random movement in the space $\mathbb{R}^d$. The trajectory of the particle is described by a d-dimensional Brownian motion B_s. Independently, there is a family of the obstacles randomly located in the space $\mathbb{R}^d$. Assume that each obstacle has mass 1 and that the obstacles are distributed in $\mathbb{R}^d$ according to a Poisson field $\omega(dx)$ with the Lebesgue measure dx as its intensity measure. Throughout, the notation "$\mathbb{P}$" and "$\mathbb{E}$" are used for the probability law and the expectation, respectively, generated by the Poisson field $\omega(dx)$, while the notation "$\mathbb{P}_x$" and "$\mathbb{E}_x$" are for the probability law and the expectation, respectively, of the Brownian motion B_s with $B_0 = x$.

The model of Brownian motion in Poisson potential has been introduced to describe the trajectory of a Brownian particle that survived being trapped by the obstacles. We refer the reader to the book by Sznitman [24] and the survey [21] made by Komorowski for a systematic account of this model and the monograph by Harvlin and Ben Avraham [20] for physicists' views on the trapping kinetics. In the usual set-up, the random field (known as potential function)

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy) \tag{1.1}$$

Received September 2010; revised January 2011.
[1] Supported in part by NSF Grant DMS-07-04024.
MSC2010 subject classifications. 60J65, 60K37, 60K40, 60G55, 60F10.
Key words and phrases. Renormalization, Poisson field, Brownian motion in Poisson potential, parabolic Anderson model, Feynman–Kac representation, large deviations.

represents the total trapping energy at $x \in \mathbb{R}^d$ generated by the Poisson obstacles, where $K(x) \geq 0$ is a deterministic function on $\mathbb{R}^d$ known as the shape function. In the quenched setting, where the observation of the system is conditioned on the environment generated by the Poisson obstacles, the model of Brownian motion in Poisson potential is often introduced as the random Gibbs measure $\mu_{t,\omega}$ on $C\{[0,t]; \mathbb{R}^d\}$ defined as

$$\frac{d\mu_{t,\omega}}{d\mathbb{P}_0} = \frac{1}{Z_{t,\omega}} \exp\left\{-\theta \int_0^t V(B_s)\,ds\right\}. \tag{1.2}$$

The integral

$$\int_0^t V(B_s)\,ds$$

measures the total trapping energy received by the Brownian particle up to the time t. Under the law $\mu_{t,\omega}$, therefore, the Brownian paths heavily impacted by the Poisson obstacles are penalized and become less likely.

Sznitman [24] considers two kinds of shape functions. In one case $K(x) = \infty 1_C$ for a nonpolar set $C \subset \mathbb{R}^d$, while in another case, the shape function $K(x)$ is assumed to be bounded and compactly supported. The correspondent potential functions are called hard and soft obstacles, respectively. In the case of hard obstacles, the Brownian particle is completely free from the influence of the obstacles until hitting the C-neighborhood of the Poisson cloud which serves as the death trap. In the setting of the soft obstacles, only the obstacles in a local neighborhood of the Brownian particle act on the particle, and the collision does not create extreme impact.

According to Newton's law of universal attraction, for example, the integrals

$$\int_{\mathbb{R}^d} \frac{1}{|y-x|^{d-1}}\omega(dy) \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{1}{|y-x|^{d-2}}\omega(dy), \qquad x \in \mathbb{R}^d,$$

represent (up to constant multiples), respectively, the total gravitational force and the total gravitational potential at the location x in the gravitational field generated by the Poisson obstacles in the case when $d \geq 3$. Therefore, it makes sense in physics to consider the shape function of the form

$$K(x) = |x|^{-p}, \qquad x \in \mathbb{R}^d. \tag{1.3}$$

A serious problem is that under choice (1.3), $V(x)$ blows up at every $x \in \mathbb{R}^d$ when $p \leq d$. In a recent paper [9], a renormalized model has been proposed as follows: First, it is shown ([9], Corollary 1.3) that under the assumption $d/2 < p < d$ the renormalized potential

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}[\omega(dy) - dy], \qquad x \in \mathbb{R}^d, \tag{1.4}$$

can be properly defined and that for any $\theta > 0$ and $t > 0$,

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{-\theta \int_0^t \overline{V}(B_s)\, dx\right\} < \infty.$$

Consequently,

$$\overline{Z}_{t,\omega} \equiv \mathbb{E}_0 \exp\left\{-\theta \int_0^t \overline{V}(B_s)\, dx\right\} < \infty \qquad \text{a.s.} \tag{1.5}$$

Thus, the Gibbs measure $\overline{\mu}_{t,\omega}$ given as

$$\frac{d\overline{\mu}_{t,\omega}}{d\mathbb{P}_0} = \frac{1}{\overline{Z}_{t,\omega}} \exp\left\{-\theta \int_0^t \overline{V}(B_s)\, ds\right\} \tag{1.6}$$

is well defined and appears to be a natural extension of $\mu_{t,\omega}$ [given in (1.2)] in the following sense: When $K(x)$ is compactly supported and bounded, by translation invariance of Lebesgue measure,

$$\begin{aligned}\overline{V}(x) &= \int_{\mathbb{R}^d} K(y-x)[\omega(dy) - dy] = \int_{\mathbb{R}^d} K(y-x)\omega(dy) - \int_{\mathbb{R}^d} K(y)\, dy \\ &= V(x) - \text{constant}.\end{aligned}$$

So the Gibbs measures generated by $V(x)$ and by $\overline{V}(x)$ are equal. We call the random path under the law $\mu_{t,\omega}$ the Brownian motion of the renormalized Poisson potential $\overline{V}(x)$. In the case when $K(x)$ is given in (1.3), the renormalized Poisson potential $\overline{V}(x)$ in (1.4) appears as the constant multiple of the Riesz potential of the compensated Poisson field $\omega(dy) - dy$.

One of major objectives of this paper is to investigate the large-t asymptotics for partition function $\overline{Z}_{t,\omega}$ given in (1.5) with the potential function $\overline{V}(x)$ be defined in (1.4).

This problem is also motivated by the parabolic Anderson formulated in the form of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \kappa \Delta u(t,x) + \xi(x) u(t,x), \\ u(0,x) = 1, \end{cases} \tag{1.7}$$

where $\kappa > 0$ is a constant called diffusion coefficient, and $\xi(x)$ is a properly chosen random field called potential.

Among other things, the parabolic Anderson models are used to describe evolution of the mass density $u(t,x)$ distributed in $\mathbb{R}^d$ (see, e.g., [9] for the discussion on this link). The mathematical relevance of the parabolic Anderson models to our topic is based on two facts: First, by the space homogeneity of the Poisson field,

$$\{\xi(t,x); t \ge 0\} \stackrel{d}{=} \{\xi(t,0); t \ge 0\}, \qquad x \in \mathbb{R}^d. \tag{1.8}$$

Consequently, the focus of the investigation is often on $u(t,0)$. Second, by the Feynman–Kac representation,

$$u(t,0) = \mathbb{E}_0 \exp\left\{\int_0^t \xi(B_{2\kappa s})\, ds\right\} = \mathbb{E}_0 \exp\left\{(2\kappa)^{-1} \int_0^{2\kappa t} \xi(B_s)\, ds\right\} \tag{1.9}$$

for sufficiently nice $\xi(x)$.

There are long lists of publications on this model among which we refer the reader to the monograph [5] by Carmona and Molchanov for the overview and background of this subject. In the usual set-up, $\xi(x) = \pm V(x)$ with $V(x)$ being given in (1.1). In the existing literature, the shape function $K(x)$ is usually assumed to be bounded and compactly supported so that the potential function $V(x)$ can be defined. A localized shape is analogous to the usual set-up in the discrete parabolic Anderson model, where the potential $\{V(x); x \in \mathbb{Z}^d\}$ is an i.i.d. sequence. On the other hand, there are practical needs for considering the cases, such that when $K(x) = |x|^{-p}$, where the environment has a long-range dependency and the extreme force surges at the locations of the Poisson obstacles.

In this paper, we consider the case when $\xi(x) = \pm\theta\overline{V}(x)$ where $\overline{V}(x)$ is defined in (1.4). Given the fact (Proposition 2.7 in [9]) that $\overline{V}(x)$ is unbounded in any neighborhood with positive probability, it is unlikely that equation (1.7) is solvable in the path-wise sense. On the other hand, it has been proved in [9] that $u(t,x)$ represented by the Feynman–Kac formula is a mild solution to (1.7) [with $\xi(x) = \pm\theta\overline{V}(x)$] whenever the quenched moment in (1.9) is finite.

The objects of our investigation are the quenched exponential moments

$$\mathbb{E}_0 \exp\left\{-\theta \int_0^t \overline{V}(B_s)\,ds\right\} \quad \text{and} \quad \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)\,ds\right\}. \tag{1.10}$$

According to (1.5), the first exponential moment in (1.10) is almost surely defined. As for the second exponential moment, it has been proved in recent work [9] that the correspondent annealed exponential moment blows up, and that, for any $\theta > 0$ and $t > 0$,

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)\,ds\right\} \begin{cases} < \infty, & \text{if } p < 2, \\ = \infty, & \text{if } p > 2. \end{cases} \tag{1.11}$$

The critical case $p = 2$, in which $d = 3$ by the constraint $d/2 < p < d$, has been investigated in a more recent paper [10] where it is shown that for any $t > 0$

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)\,ds\right\} \begin{cases} < \infty, & \text{a.s. when } \theta < \frac{1}{16}, \\ = \infty, & \text{a.s. when } \theta > \frac{1}{16}. \end{cases} \tag{1.12}$$

The main objective of this paper is to investigate the quenched large-t asymptotics for the exponential moments given in (1.10) whenever these moments are finite, except the critical case described in (1.12) (which is studied in [10]). We point out the references [3–5, 7, 11–14, 17, 18, 22–24, 26, 27] as an incomplete list related to this topic.

For later comparison, we mention some existing results which are narrowly relevant to the topic of this paper. Let the potential function $V(x)$ be given in (1.1). Sznitman ([24], Theorem 5.3, page 196) shows that for the bounded and compactly supported shape $K(\cdot)$ and $\theta > 0$,

$$\lim_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log \mathbb{E}_0 \exp\left\{-\theta \int_0^t V(B_s)\,ds\right\} = -\lambda_d \left(\frac{\omega_d}{d}\right)^{2/d} \qquad \text{a.s.-}\mathbb{P}, \tag{1.13}$$

where $\lambda_d > 0$ is the principal eigenvalue of the Laplacian operator $(1/2)\Delta$ on the d-dimensional unit ball with zero boundary values, and ω_d is the volume of the d-dimensional unit ball. With a slightly different formulation [22], the model of hard obstacles yields the same pattern of asymptotics.

Under some continuity, boundedness assumptions on $K(x)$ and under some restriction on the tail of $K(x)$, Carmona and Molchanov ([6], Theorem 5.1) prove that

$$\lim_{t\to\infty} \frac{\log\log t}{t\log t}\log \mathbb{E}_0 \exp\left\{\theta\int_0^t V(B_s)\,ds\right\} = d\theta \sup_{x\in\mathbb{R}^d} K(x) \qquad \text{a.s.-}\mathbb{P}. \tag{1.14}$$

The interested reader is also referred to [19] and [17] for the correspondent asymtotics of the second order.

After the first draft of this paper was completed, the author learned the recent investigation by Fukushima [15] in the case when $K(x) = |x|^{-p}\wedge 1$ with $d < p < d+2$, the setting where no renormalization is necessary. Fukushima [15] shows that

$$\begin{aligned} &\lim_{t\to\infty} t^{-1}(\log t)^{-(p-d)/d}\log \mathbb{E}_0 \exp\left\{-\int_0^t V(B_s)\,ds\right\} \\ &\qquad = -\frac{d}{p}\left(\frac{p-d}{pd}\right)^{(p-d)/d}\left(\omega_d\Gamma\left(\frac{p-d}{d}\right)\right)^{p/d} \qquad \text{a.s.-}\mathbb{P}. \end{aligned} \tag{1.15}$$

It should be mentioned that Fukushima also obtained the second asymptotic term in his setting.

2. Main theorems and strategies.

Throughout this paper, let ω_d be the volume of the d-dimensional unit ball. Let $W^{1,2}(\mathbb{R}^d)$ denote the Sobolev space given as

$$W^{1,2}(\mathbb{R}^d) = \{f\in\mathcal{L}^2(\mathbb{R}^d);\ \nabla f\in\mathcal{L}^2(\mathbb{R}^d)\}.$$

By (A.4) below, when $d/2 < p < \min\{d,2\}$ there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^d}\frac{f^2(x)}{|x|^p}\,dx \le C\|f\|_2^{2-p}\|\nabla f\|_2^p, \qquad f\in W^{1,2}(\mathbb{R}^d).$$

Let $\sigma(d,p) > 0$ be the best constant in above inequality.

The main theorems are stated as follows.

THEOREM 2.1. *Under $d/2 < p < d$,*

$$\begin{aligned} &\lim_{t\to\infty} t^{-1}(\log t)^{-(d-p)/d}\log \mathbb{E}_0 \exp\left\{-\theta\int_0^t \overline{V}(B_s)\,ds\right\} \\ &\qquad = \frac{\theta d^2}{d-p}\left(\frac{\omega_d}{d}\Gamma\left(\frac{2p-d}{p}\right)\right)^{p/d} \qquad a.s.\text{-}\mathbb{P} \end{aligned} \tag{2.1}$$

for every $\theta > 0$.

THEOREM 2.2. *Under $d/2 < p < \min\{2, d\}$,*

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\left(\frac{\log\log t}{\log t}\right)^{2/(2-p)}&\log\mathbb{E}_0\exp\left\{\theta\int_0^t\overline{V}(B_s)\,ds\right\}\\ &=\frac{1}{2}p^{p/(2-p)}(2-p)^{(4-p)/(2-p)}\left(\frac{d\theta\,\sigma(d,p)}{2+d-p}\right)^{2/(2-p)}\qquad a.s.\text{-}\mathbb{P}\end{aligned}\tag{2.2}$$

for every $\theta > 0$.

We now make a comparison of "(1.13) versus (2.1)" and "(1.14) versus (2.2)." First, the quenched exponential moments in our models generate significantly larger quantities. Second, a heavy shape dependence (or p-dependence) presented in our theorems sharply contrasts the shape insensitivity appearing in (1.13) and (1.14). In Theorem 2.1, it is the nonlocality of the shape function that plays a major role, while the high peaks of $\overline{V}(x)$ correspond to small values of the quenched exponential moment. On the other hand, the asymptotics in Theorem 2.2 is shaped by the singularity of $K(x) = |x|^{-p}$ at $x = 0$. In addition, there seems to be a degree of resemblance between (1.15) and (2.1). Based on the comment made about roles of nonlocality and singularity, it may be possible that (1.15) remains true even without removing the singularity of $K(x)$ at $x = 0$. We leave this problem to future study.

Does the Lebesgue measure in renormalization contribute to the limit laws stated in Theorems 2.1 and 2.2? The answer is "Yes" to Theorem 2.1, for otherwise the right-hand side of (2.1) would be negative. The answer is "No" to Theorem 2.2 as the major impact comes from the Poisson points in a very small neighborhood of the site where the Brownian particle is located [see (2.12) below for a more quantified analysis on this point].

Associated with the spatial Brownian motion in the classic gravitational field generated by the Poisson obstacles, the following corollary appears as Theorem 2.1 in the special case $d = 3$ and $p = 2$.

COROLLARY 2.3. *When $d = 3$ and $p = 2$,*

$$\lim_{t\to\infty} t^{-1}(\log t)^{-1/3}\log\mathbb{E}_0\exp\left\{-\theta\int_0^t\overline{V}(B_s)\,ds\right\} = 3\sqrt[3]{12}\pi\theta\qquad a.s.\text{-}\mathbb{P}\tag{2.3}$$

for every $\theta > 0$.

Let $u_0(t,x)$ and $u_1(t,x)$ be the mild solutions to the parabolic Anderson problems (1.7) that satisfy the Feynman–Kac representation (1.9) with $\xi(x) = -\theta\overline{V}(x)$ and $\xi(x) = \theta\overline{V}(x)$, respectively. By the space homogeneity (1.8) and by Theorems 2.1 and 2.2,

$$\begin{aligned}\lim_{t\to\infty}&\, t^{-1}(\log t)^{-(d-p)/d}\log u_0(t,x)\\ &=\frac{\theta d^2}{d-p}\left(\frac{\omega_d}{d}\Gamma\left(\frac{2p-d}{p}\right)\right)^{p/d}\qquad \text{a.s.-}\mathbb{P},\end{aligned}\tag{2.4}$$

$$\lim_{t\to\infty}\frac{1}{t}\left(\frac{\log\log t}{\log t}\right)^{2/(2-p)}\log u_1(t,x) \tag{2.5}$$
$$=\frac{1}{2}\left(\frac{p}{2\kappa}\right)^{p/(2-p)}(2-p)^{(4-p)/(2-p)}\left(\frac{d\theta\,\sigma(d,p)}{2+d-p}\right)^{2/(2-p)}\qquad \text{a.s.-}\mathbb{P}$$

for every $\theta > 0$ and $x \in \mathbb{R}^d$.

An immediate observation is that the diffusion coefficient κ does not appear in (2.4). The same phenomena have been noticed by Carmona and Molchanov [6] in the case when $\xi(x) = \theta V(x)$ for the same $V(x)$ appearing in (1.14).

In the following we compare the strategies for the laws given in (1.13), (1.14), (2.1) and (2.2). To make the discussion more informative, we focus on the lower bounds and try to describe the behavior of the Brownian particle and the behavior of the Poisson particle in each strategy. The treatment for (1.13) and (1.14) does not have to be the same as their original proof. In our discussion, we use the notation $B(x, R)$ for the d-dimensional ball of the center x and radius R.

The following ingredients on the behavior of the Brownian particle are common to all strategies: Up to the time t the Brownian particle stays in the ball $B(0, R_t)$ (referred as "macro-ball") with the radius R_t roughly equal to t.[2] Within a period $[0, ut]$ (with a very small $u > 0$), the Brownian particle moves into one of the roughly t^d prearranged and evenly located identical micro-balls

$$D_z \equiv B(z, r_t); \qquad z \in b_t\mathbb{Z}^d \cap B(0, R_t), \tag{2.6}$$

where $r_t \ll b_t$ and $r_t R_t \ll t$. The principle that Brownian particle chooses D_z is to maximize the positive energy (or to minimize the negative energy) from the Poisson field.

The main difference among different strategies in the Brownian path is on the radius r_t of the microbes. By the relation $r_t R_t \ll t$ and by a classic small ball estimate, the cost for the Brownian particle to choose D_z is ($\delta > 0$ is a small number here)

$$\mathbb{P}_0\{\text{The Brownian particle reaches } D_z \text{ quickly and then stays in } D_z \text{ up to } t\}$$
$$\geq \frac{1}{(2\pi)^d}\int_{B(z,\delta r_t)} e^{-|x|^2/(2ut)}\mathbb{P}_0\{B_s \in B(z-x, r_t) \text{ for } 0 \leq s \leq (1-u)t\}\,dx \tag{2.7}$$
$$\approx \exp\{-o(r_t^{-2}t)\}\mathbb{P}_0\Big\{\sup_{0\leq s\leq t}|B_s| \leq r_t\Big\} \approx \exp\{-\lambda_d r_t^{-2}t\}.$$

Here we recall that $\lambda_d > 0$ is the principle eigenvalue of the Laplacian operator $(1/2)\Delta$ on the d-dimensional unit ball with zero boundary condition. To make the cost affordable compared with the deviation scale $t(\log t)^{-2/d}$ in the strategy for

[2]The combination of the word "roughly" and a big number t means $tL(t)$ with $L(t)$ slow-varying at ∞.

(1.13), for example, the radius r_t should be at least $r(\log t)^{1/d}$ with the constant $r > 0$. Based on the same principle, the critical radius of the micro-balls in each strategy are determined as following:

$$r_t = \begin{cases} r(\log t)^{1/d}, & \text{in the strategy for (1.13)}, \\ r\sqrt{\dfrac{\log\log t}{\log t}}, & \text{in the strategy for (1.14)}, \\ r(\log t)^{-(d-p)/(2d)}, & \text{in the strategy for (2.1)}, \\ r\left(\dfrac{\log\log t}{\log t}\right)^{1/(2-p)}, & \text{in the strategy for (2.2)}. \end{cases} \tag{2.8}$$

We now describe the behavior of the Poisson field in each strategy. For (1.13), the high peak of the quenched moment occurs when $\int_0^t V(B_s)\,ds \approx 0$. To make this happen, one of the C-neighborhoods $\widetilde{D}_z \equiv D_z + C$ $[z \in b_t\mathbb{Z}^d \cap B(0, R_t)]$ is obstacle-free, where $C \subset \mathbb{R}^d$ is the compact support of $K(x)$, and the Brownian particle spends most of its time in that same micro-ball D_z. In view of (2.7), therefore,

$$\mathbb{E}_0 \exp\left\{-\theta \int_0^t V(B_s)\,ds\right\} \succeq \exp\{-\lambda_d r^{-2} t(\log t)^{-2/d}\}$$

on the event $\{\min_z \omega(\widetilde{D}_z) = 0\}$, where the relation "$\succeq$" reads as "asymptotically greater than or equivalent to."

On the other hand,

$$\begin{aligned} \mathbb{P}\Big\{\min_z \omega(\widetilde{D}_z) = 0\Big\} &\approx 1 - \big(1 - \mathbb{P}\{\omega(\widetilde{D}_0) = 0\}\big)^{t^d} \\ &= 1 - (1 - \exp\{-\omega_d r^d \log t\})^{t^d} \\ &\approx 1 - \exp\{-t^{d-\omega_d r^d}\}. \end{aligned}$$

Hence, a standard way of using the Borel–Cantelli lemma shows that the phase transition between

$$\mathbb{P}\Big\{\min_z \omega(\widetilde{D}_z) = 0 \text{ eventually}\Big\} = 1 \quad \text{and} \\ \mathbb{P}\Big\{\min_z \omega(\widetilde{D}_z) \geq 1 \text{ eventually}\Big\} = 1 \tag{2.9}$$

occurs when r satisfies $\omega_d r^d = d$. Consequently, this strategy leads to the lower bound requested by (1.13).

In the strategy for (2.1), only the impact of the Poisson obstacles within the distance $a(\log t)^{1/d}$ from the Brownian particle is counted. To determine constant $a > 0$, a crucial problem is whether or not the high peak can be captured by the "empty ball" strategy which means to make the ball $B(B_s, a(\log t)^{1/d})$

[$\approx B(z, a(\log t)^{1/d})$ as the Brownian particle stays in D_z] free of the Poisson obstacles. Under the "empty-ball" strategy,

$$\int_0^t \overline{V}(B_s)\,ds \approx \int_0^t V_1(B_s)\,ds - t\int_{\{|x|\le a(\log t)^{1/d}\}} \frac{1}{|x|^p}\,dx$$
$$\approx -\frac{a^{d-p}\omega_d}{d-p} t(\log t)^{(d-p)/d},$$

where

$$V_1(x) = \int_{|y-x|\le a(\log t)^{1/d}\}} \frac{\omega(dy)}{|y-x|^p}.$$

On the other hand, the estimate given in (2.9) shows that the largest radius R for one of the balls $B(z, R)$ [$z \in b_t\mathbb{Z}^d \cap B(0, R_t)$] to be obstacle-free is $R = (\omega_d^{-1} d)^{1/d}(\log t)^{1/d}$. By making $r > 0$ sufficiently large in (2.7), the best lower bound that the "empty-ball" strategy can offer is

$$\liminf_{t\to\infty} t^{-1}(\log t)^{-(d-p)/d} \log \mathbb{E}_0 \exp\left\{-\theta \int_0^t \overline{V}(B_s)\,ds\right\}$$
$$\ge \frac{\theta}{d-p} d^{(d-p)/d}\omega_d^{p/d} \qquad \text{a.s.}$$

under the optimal choice $a = (\omega_d^{-1} d)^{1/d}$. In comparison with (2.1), this bound gives the right rate but not the right constant.

Based on the above analysis, we conclude that the constant $a > 0$ has to be arbitrarily large and that the "empty-ball" strategy is not working well for (2.1).

We now come to (1.14). By the continuity assumption on the shape function and by homogeneity of the Poisson field, the supremum $\sup_{x\in\mathbb{R}^d} K(x)$ can be achieved somewhere, and we may assume that $K(0) = \sup_{x\in\mathbb{R}^d} K(x)$ in the following discussion. To support the limit law given in (1.14), the Poisson field executes a strategy that fills one of the δ-balls $\{B(z,\delta); z \in b_t\mathbb{Z}^d \cap B(0, R_t)\}$ with a high density of the Poisson points, where the constant $\delta > 0$ is (arbitrarily) small but fixed. By translation invariance and by continuity of $K(x)$, for any $z \in b_t\mathbb{Z}^d \cap B(0, R_t)$

$$V(B_s) = \int_{\mathbb{R}^d} K(x - B_s)\omega(dx)$$
$$= \int_{\mathbb{R}^d} K\big(x - (B_s - z)\big)\omega(z + dx) \succeq K(0)\omega(B(z,\delta))$$

as $B_s \in D_z$. By (2.7), therefore,

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t V(B_s)\,ds\right\}$$
$$\succeq \exp\left\{\theta K(0) t \max_z \omega(B(z,\delta)) - \lambda_d r^{-2}\frac{t\log t}{\log\log t}\right\}. \tag{2.10}$$

On the other hand, by independence

$$\mathbb{P}\left\{\max_z \omega(B(z,\delta)) \ge \sigma \frac{\log t}{\log\log t}\right\} \approx 1 - \left(1 - \mathbb{P}\left\{\omega(B(0,\delta)) \ge \sigma \frac{\log t}{\log\log t}\right\}\right)^{t^d}$$
$$\approx 1 - (1 - \exp\{-\sigma \log t\})^{t^d}$$
$$\approx 1 - \exp\{-t^{d-\sigma}\} \qquad \forall \sigma > 0.$$

Using the Borel–Cantelli lemma we can prove that

$$\lim_{t\to\infty} \frac{\log\log t}{\log t} \max_z \omega(B(z,\delta)) = d \qquad \text{a.s.} \tag{2.11}$$

Since $r > 0$ can be arbitrarily large, (2.10) and (2.11) lead to the lower bound requested by (1.14).

The strategy that Poisson field executes in (2.2) is to fill one of the balls

$$B\left(z, \delta\left(\frac{\log\log t}{\log t}\right)^{1/(2-p)}\right); \qquad z \in b_t\mathbb{Z}^d \cap B(0, R_t),$$

with a high concentration of the Poisson points. In the following we present a simple algorithm to illustrate the idea. Assume that the Brownian particle spends most of its time in D_z for some $z \in b_t\mathbb{Z}^d \cap B(0, R_t)$. Given a fixed $a > 0$, it is not hard to show that the impact of the Poisson points which are a-unit away from the Brownian particle is negligible, and that the "renormalizer" does not make any noticeable contribution to the limit law in (2.2). Hence,

$$\begin{aligned}
\overline{V}(B_s) &= \int_{\mathbb{R}^d} \frac{1}{|x - (B_s - z)|^p}[\omega(z+dx) - dx] \\
&\approx \int_{|x-(B_s-z)|\le a\}} \frac{1}{|x-(B_s-z)|^p}\omega(z+dx) \qquad (2.12) \\
&\ge (\delta + r)^{-p}\left(\frac{\log t}{\log\log t}\right)^{p/(2-p)} \omega\left\{x; |z+x| \le \delta\left(\frac{\log\log t}{\log t}\right)^{1/(2-p)}\right\}.
\end{aligned}$$

Write

$$X_z = \omega\left\{y; |z+y| \le \delta\left(\frac{\log\log t}{\log t}\right)^{1/(2-p)}\right\}.$$

In view of (2.7),

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)\,ds\right\} \\
&\quad \succeq \exp\left\{(r+\delta)^{-p}\theta t\left(\frac{\log t}{\log\log t}\right)^{p/(2-p)} \max_z X_z \right.\\
&\qquad\qquad \left. - \lambda_d r^{-2} t\left(\frac{\log t}{\log\log t}\right)^{2/(2-p)}\right\}.
\end{aligned}$$

Similarly to (2.11),

$$\lim_{t\to\infty} \frac{\log\log t}{\log t} \max_z X_z = \frac{d(2-p)}{3-p} \qquad \text{a.s.}$$

Since $\delta > 0$ can be arbitrarily small, the optimal pick

$$r = \left(\frac{2\lambda_d(3-p)}{dp(2-p)\theta}\right)^{1/(2-p)}$$

leads to the lower bound

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t}\left(\frac{\log\log t}{\log t}\right)^{2/(2-p)} \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)\,ds\right\} \\
&\qquad \geq \frac{1}{2}\left(\frac{p}{2\lambda_d}\right)^{p/(2-p)} (2-p)^{(4-p)/(2-p)} \left(\frac{d\theta}{3-p}\right)^{2/(2-p)} \qquad \text{a.s.}
\end{aligned}$$

This bound is sharp in rate in comparison with (2.2). Due to a lack of information on the value of $\sigma(d,p)$, we are not able to compare the constants on the right-hand sides. However, it looks unlikely that the constant obtained here would match the one in (2.2). In addition, the argument given in Sections 5 and 6 shows that the constant $r > 0$ should be arbitrarily large for the accuracy requested by (2.2).

In summary, the simple strategies given above provide some heuristic pictures on the behavior patterns of both Brownian particles and the Poisson field and can be made rigorous for (1.13) and (1.14), but fall short of the accuracy demanded by (2.1) and (2.2). Some harder computation on the tail estimates for Poisson integrals is needed for the main theorems in this paper.

We now comment on the methods used in this paper. The Feynman–Kac formula is essential in this paper for tracking the principal eigenvalues. Among others, the ingenious approach developed in [16] and [17], which allows one to bound the principal eigenvalue over a large domain by the maximal of the principal eigenvalues over the sub-domains, plays a key role in our argument for the upper bound. With this approach, we reduce the problem essentially to the tail estimate of the random Dirichlet form

$$\sup_{g\in\mathcal{F}_d(B(0,r\varepsilon^{1/d}))} \left\{\pm\theta \int_{B(0,r\varepsilon^{1/d})} \overline{V}(x)g^2(x)\,dx - \frac{1}{2}\int_{B(0,r\varepsilon^{1/d})} |\nabla g(x)|^2\,dx\right\}, \tag{2.13}$$

where for any domain $D \subset \mathbb{R}^d$, $\mathcal{F}_d(D)$ is defined as the set of the smooth functions g on D with $\|g\|_{\mathcal{L}^2(D)} = 1$ and $g(\partial D) = 0$, the constant $r > 0$ is large but fixed and associated with the critical radius r_t posted in (2.8), the parameter ε is given as follows:

$$\varepsilon = \begin{cases} (\log t)^{-(d-p)/2}, & \text{in the proof of Theorem 2.1,} \\ \left(\dfrac{\log\log t}{\log t}\right)^{d/(2-p)}, & \text{in the proof of Theorem 2.2.} \end{cases} \tag{2.14}$$

Another important idea adopted in this paper is the Poisson field rescaling. In his proof of (1.13), Sznitman ([24], Chapter 4) reduces the problem to the investigation of the "enlarged obstacles"

$$\omega((\log t)^{1/d}\,dx).$$

It is worth pointing out that the choice of the rescaling factor $(\log t)^{1/d}$ links to the critical radius r_t posted in (2.8). What we confront here are the "contracted obstacles" $\omega(\varepsilon\,dx) - \varepsilon\,dx$ with $\varepsilon > 0$ given in (2.14). Under the substitution $g(x) \mapsto \varepsilon^{-1/2} g(\varepsilon^{-1/d} x)$ and by Fubini's theorem, the variation in (2.13) is equal to

$$\sup_{g\in\mathcal{F}_d(B(0,r))}\left\{\pm\theta\varepsilon^{-p/d}\int_{\mathbb{R}^d}\left[\int_{B(0,r)}\frac{g^2(y)}{|y-x|^p}\,dy\right][\omega(\varepsilon\,dx)-\varepsilon\,dx]\right.$$
$$\left. - \frac{\varepsilon^{-2/d}}{2}\int_{B(0,r)}|\nabla g(x)|^2\,dx\right\}.$$

The tail probabilities of the compensated Poisson integral appearing here will be the main topic of the next section.

In comparison to the existing literature such as [2, 17, 25, 26], perhaps the most substantial difference comes from the fact that in these works, the logarithmic moment generating function (or the fractional logarithmic moment generating function)

$$H(\gamma) \equiv \log\mathbb{E}\exp\{\gamma V(0)\}$$

exists. As a matter of fact, it is the logarithmic moment generating function $H(\gamma)$ (or the fractional logarithmic moment generating function) that plays a decisive role in these publications in determining the asymptotics for

$$\log\mathbb{E}_0\exp\left\{\theta\int_0^t V(B_s)\,ds\right\} \quad \text{and}$$
$$\log\mathbb{E}\otimes\mathbb{E}_0\exp\left\{\theta\int_0^t V(B_s)\,ds\right\} \qquad (t\to\infty)$$

through some well-developed algorithms. Unfortunately, this is not our case. Indeed, we have that

$$\mathbb{E}\overline{V}^2(0) = \int_{\mathbb{R}^d}\frac{dx}{|x|^{2p}} = \infty.$$

Additional challenges we confront are the local unboundedness of $\overline{V}(x)$, and the loss of monotonicity of Poisson integrals due to renormalization.

The rest of the paper is organized as follows. In Section 3 we establish the large deviations for a group of Poisson integrals with respect to the contracted renormalized Poisson field. In Section 4, some explicit bounds for the Feynman–Kac formula are established for later application. The upper bounds and the lower

bounds for our main theorems are proved in Sections 5 and 6, respectively. These bounds are established simultaneously for Theorems 2.1 and 2.2. In Section 6, some identities for the relevant integrals and variations are established.

3. Large deviations for Poisson integrals. The functions

$$\psi(\lambda) = e^{-\lambda} - 1 + \lambda \quad \text{and} \quad \Psi(\lambda) = e^{\lambda} - 1 - \lambda \qquad (\lambda \geq 0)$$

appear frequently in this section. It is easy to see that $\psi(\lambda)$ and $\Psi(\lambda)$ are non-negative, increasing and convex on $[0, \infty)$ with $\psi(0) = \Psi(0) = 0$. In addition, $\psi(\cdot) \leq \Psi(\cdot)$ on $[0, \infty)$. According to Lemma A.1,

$$\int_{\mathbb{R}^d} \psi\left(\frac{1}{|x|^p}\right) dx = \omega_d \frac{p}{d-p} \Gamma\left(\frac{2p-d}{p}\right) \tag{3.1}$$

when $d/2 < p < d$.

The function $\Psi(|\cdot|^{-p})$ is not integrable on $\mathbb{R}^d$. Under $p > d/2$, however,

$$\int_{\{|x| \geq c\}} \Psi\left(\frac{1}{|x|^p}\right) dx < \infty, \qquad c > 0.$$

Throughout this section, $D \subset \mathbb{R}^d$ is a fixed bounded open set. Write

$$\mathcal{G}_d(D) = \left\{g \in W^{1,2}(D); \|g\|_{\mathcal{L}^2(D)}^2 + \tfrac{1}{2}\|\nabla g\|_{\mathcal{L}^2(D)}^2 = 1\right\}, \tag{3.2}$$

where $W^{1,2}(D)$ is the Sobolev space over D, defined to be the closure of the inner product space consisting of the infinitely differentiable functions compactly supported in D under the Sobolev norm

$$\|g\|_H = \left\{\|g\|_{\mathcal{L}^2(D)}^2 + \|\nabla g\|_{\mathcal{L}^2(D)}^2\right\}^{1/2}.$$

To reserve continuity we adopt a smooth truncation to the shape function. Let the smooth function $\alpha : \mathbb{R}^+ \longrightarrow [0, 1]$ satisfy the following properties: $\alpha(\lambda) = 1$ on $[0, 1]$, $\alpha(\lambda) = 0$ for $\lambda \geq 3$ and $-1 \leq \alpha'(\lambda) \leq 0$.

For $a > 0$ and $\varepsilon > 0$, define

$$K_{a,\varepsilon}^{(0)}(x) = \frac{1}{|x|^p} \alpha\left(a^{-1} \varepsilon^{(2+d-p)/(d(d-p))} |x|\right),$$

$$K_{a,\varepsilon}^{(1)}(x) = \frac{1}{|x|^p} \alpha(a^{-1} (\log \varepsilon^{-1})^{-1/p} |x|)$$

and

$$L_{a,\varepsilon}^{(0)}(x) = \frac{1}{|x|^p} \left\{1 - \alpha\left(a^{-1} \varepsilon^{(2+d-p)/(d(d-p))} |x|\right)\right\},$$

$$L_{a,\varepsilon}^{(1)}(x) = \frac{1}{|x|^p} \{1 - \alpha(a^{-1} (\log \varepsilon^{-1})^{-1/p} |x|)\}$$

and

$$G^{(i)}_{a,\varepsilon}(g)=\int_{\mathbb{R}^d}\biggl[\int_D K^{(i)}_{a,\varepsilon}(y-x)g^2(y)\,dy\biggr][\omega(\varepsilon\,dx)-\varepsilon\,dx], \qquad g\in\mathcal{G}_d(D),\ i=0,1, \tag{3.3}$$

$$F^{(i)}_{a,\varepsilon}(g)=\int_{\mathbb{R}^d}\biggl[\int_D L^{(i)}_{a,\varepsilon}(y-x)g^2(y)\,dy\biggr][\omega(\varepsilon\,dx)-\varepsilon\,dx], \qquad g\in\mathcal{G}_d(D),\ i=0,1. \tag{3.4}$$

Write

$$\zeta_\varepsilon(g)=\int_{\mathbb{R}^d}\biggl[\int_D \frac{g^2(y)}{|y-x|^p}\,dy\biggr][\omega(\varepsilon\,dx)-\varepsilon\,dx], \qquad g\in\mathcal{G}_d(D). \tag{3.5}$$

The main theorems in this section are the large deviations for the Poisson integrals indexed by $\mathcal{G}_d(D)$.

THEOREM 3.1. *Assume that $d/2<p<d$. For any $a>0$ and $\gamma>0$,*

$$\lim_{a\to\infty}\limsup_{\varepsilon\to 0^+}\varepsilon^{2/(d-p)}\log\mathbb{P}\Bigl\{\sup_{g\in\mathcal{G}_d(D)}\bigl|F^{(0)}_{a,\varepsilon}(g)\bigr|\ge\gamma\varepsilon^{-(2-p)/d}\Bigr\}=-\infty, \tag{3.6}$$

$$\liminf_{a\to\infty}\liminf_{\varepsilon\to 0^+}\varepsilon^{2/(d-p)}\log\mathbb{P}\Bigl\{\inf_{g\in\mathcal{G}_d(D)}G^{(0)}_{a,\varepsilon}(g)\le-\gamma\varepsilon^{-(2-p)/d}\Bigr\}\ge-I_D(\gamma), \tag{3.7}$$

$$\lim_{\varepsilon\to 0^+}\varepsilon^{2/(d-p)}\log\mathbb{P}\Bigl\{\inf_{g\in\mathcal{G}_d(D)}\zeta_\varepsilon(g)\le-\gamma\varepsilon^{-(2-p)/d}\Bigr\}=-I_D(\gamma), \tag{3.8}$$

where

$$I_D(\gamma)=\biggl(\frac{\gamma(d-p)}{d}\biggr)^{d/(d-p)}\biggl(\omega_d\Gamma\biggl(\frac{2p-d}{p}\biggr)\biggr)^{-p/(d-p)} \times\Bigl(\sup_{g\in\mathcal{G}_d(D)}\|g\|_{\mathcal{L}^2(D)}\Bigr)^{-2d/(d-p)}. \tag{3.9}$$

Write $l(\varepsilon)=\varepsilon^{-(2-p)/d}\log\frac{1}{\varepsilon}$, and

$$\rho^*_D=\sup_{g\in\mathcal{G}_d(D)}\sup_{x\in\mathbb{R}^d}\int_D\frac{g^2(y)}{|y-x|^p}\,dy. \tag{3.10}$$

The finiteness of ρ^*_D can be seen from (A.6) in Lemma A.3 and from (A.9).

THEOREM 3.2. *Assume $d/2<p<\min\{2,d\}$. For any $a>0$ and $\gamma>0$,*

$$\lim_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{P}\Bigl\{\sup_{g\in\mathcal{G}_d(D)}\bigl|F^{(1)}_{a,\varepsilon}(g)\bigr|\ge\gamma\varepsilon^{-(2-p)/d}\Bigr\}=-\infty, \tag{3.11}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(D)} G^{(1)}_{a,\varepsilon}(g)\ge\gamma\varepsilon^{-(2-p)/d}\Big\}=-\frac{2+d-p}{d\rho_D^*}\gamma, \tag{3.12}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(D)} \zeta_\varepsilon(g)\ge\gamma\varepsilon^{-(2-p)/d}\Big\}=-\frac{2+d-p}{d\rho_D^*}\gamma. \tag{3.13}$$

Write

$$\overline{V}^{(i)}_{a,\varepsilon}(x)=\int_{\mathbb{R}^d} L^{(i)}_{a,\varepsilon}(y-x)[\omega(\varepsilon\,dy)-\varepsilon\,dy],\qquad x\in D, i=0,1.$$

Our approach relies on the following lemma.

LEMMA 3.3. *For any $a>0$ and $\theta>0$,*

$$\begin{aligned}\lim_{\varepsilon\to 0^+}&\varepsilon^{2/(d-p)}\log\mathbb{E}\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)\Big\}\\&=\int_{\mathbb{R}^d}\psi\Big(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\Big)\,dx,\end{aligned}\tag{3.14}$$

$$\begin{aligned}\lim_{\varepsilon\to 0^+}&\varepsilon^{2/(d-p)}\log\mathbb{E}\exp\Big\{\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\sup_{x\in D}\big|\overline{V}^{(0)}_{a,\varepsilon}(x)\big|\Big\}\\&=\int_{\mathbb{R}^d}\Psi\Big(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\Big)\,dx,\end{aligned}\tag{3.15}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{E}\exp\Big\{\theta\Big(\log\frac{1}{\varepsilon}\Big)\sup_{x\in D}\big|\overline{V}^{(1)}_{a,\varepsilon}(x)\big|\Big\}=0. \tag{3.16}$$

PROOF. Notice that for any $a>0$, $\theta>0$ and $x\in D$,

$$\begin{aligned}&\mathbb{E}\exp\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\overline{V}^{(0)}_{a,\varepsilon}(x)\}\\&\quad=\mathbb{E}\exp\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\overline{V}^{(0)}_{a,\varepsilon}(0)\}\\&\quad=\exp\Big\{\varepsilon\int_{\mathbb{R}^d}\psi\big(\theta\varepsilon^{-p(2+d-p)/(d(d-p))}L^{(0)}_{a,\varepsilon}(x)\big)\,dx\Big\}\\&\quad=\exp\Big\{\varepsilon^{2/(d-p)}\int_{\mathbb{R}^d}\psi\Big(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\Big)\,dx\Big\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\exp\{\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\overline{V}^{(0)}_{a,\varepsilon}(x)\}\\&\quad=\exp\Big\{\varepsilon^{2/(d-p)}\int_{\mathbb{R}^d}\Psi\Big(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\Big)\,dx\Big\}.\end{aligned}$$

In view of the fact that $\psi(\cdot) \le \Psi(\cdot)$ on $[0, \infty)$, we conclude that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\overline{V}^{(0)}_{a,\varepsilon}(x)\} = \int_{\mathbb{R}^d} \psi\left(\theta \frac{1-\alpha(a^{-1}|x|)}{|x|^p}\right) dx, \tag{3.17}$$

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\{\theta\varepsilon^{-p(2+d-p)/(d(d-p))}|\overline{V}^{(0)}_{a,\varepsilon}(x)|\} = \int_{\mathbb{R}^d} \Psi\left(\theta \frac{1-\alpha(a^{-1}|x|)}{|x|^p}\right) dx. \tag{3.18}$$

A similar computation also leads to

$$\lim_{\varepsilon\to 0^+} \frac{1}{l(\varepsilon)} \log \mathbb{E}\exp\left\{\theta\left(\log\frac{1}{\varepsilon}\right)|\overline{V}^{(1)}_{a,\varepsilon}(x)|\right\} = 0, \qquad x \in D. \tag{3.19}$$

All we need is to take supremum over $x \in D$ in the exponent on the left-hand sides of (3.17), (3.18) and (3.19) and push the supremum through the expectation. Due to similarity, we only carry out this algorithm to (3.17) and (3.18).

By the boundedness of D, we may assume that $D = (-b, b)^d$ for some $b > 0$. Let $h > 0$ be a constant which will be later specified, and let

$$\gamma = \frac{2+d-p}{d-p} + h.$$

By integration substitution

$$\begin{aligned}\overline{V}^{(0)}_{a,\varepsilon}(x) &= \varepsilon^{-ph/d}\int_{\mathbb{R}^d} \widetilde{L}^{(0)}_{a,\varepsilon}(y - \varepsilon^{-h/d}x)[\omega(\varepsilon^{1+h}\,dy) - \varepsilon^{1+h}\,dy] \\ &= \varepsilon^{-ph/d} H_\varepsilon(\varepsilon^{-h/d}x),\end{aligned} \tag{3.20}$$

where

$$\widetilde{L}^{(0)}_{a,\varepsilon}(x) = \frac{1}{|x|^p}\{1 - \alpha(a^{-1}\varepsilon^{\gamma/d}|x|)\}$$

and

$$H_\varepsilon(x) = \int_{\mathbb{R}^d} \widetilde{L}^{(0)}_{a,\varepsilon}(z - x)[\omega(\varepsilon^{1+h}\,dz) - \varepsilon^{1+h}\,dz].$$

For any $x, y \in D$ with $x \ne y$, and $\theta > 0$,

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\varepsilon^{-p\gamma/d}\frac{H_\varepsilon(x) - H_\varepsilon(y)}{|x-y|}\right\} \\ &\quad = \exp\left\{\varepsilon^{1+h}\int_{\mathbb{R}^d} \Psi\left(\varepsilon^{-p\gamma/d}\frac{\theta}{|x-y|}(\widetilde{L}^{(0)}_{a,\varepsilon}(z-x) - \widetilde{L}^{(0)}_{a,\varepsilon}(z-y))\right) dz\right\}.\end{aligned}$$

Switching x and y, one has

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\varepsilon^{-p\gamma/d}\frac{|H_\varepsilon(x)-H_\varepsilon(y)|}{|x-y|}\right\}\\&\quad\le 2\exp\left\{\varepsilon^{1+h}\int_{\mathbb{R}^d}\Psi\left(\varepsilon^{-p\gamma/d}\frac{\theta}{|x-y|}\big|\widetilde{L}^{(0)}_{a,\varepsilon}(z-x)-\widetilde{L}^{(0)}_{a,\varepsilon}(z-y)\big|\right)dz\right\}.\end{aligned}$$

By integration substitution,

$$\begin{aligned}&\int_{\mathbb{R}^d}\Psi\left(\varepsilon^{-p\gamma/d}\frac{\theta}{|x-y|}\big|\widetilde{L}^{(0)}_{a,\varepsilon}(z-x)-\widetilde{L}^{(0)}_{a,\varepsilon}(z-y)\big|\right)dz\\&\qquad=\varepsilon^{-\gamma}\int_{\mathbb{R}^d}\Psi\left(\frac{\theta}{|x-y|}|L_a(z-\varepsilon^{\gamma/d}x)-L_a(z-\varepsilon^{\gamma/d}y)|\right)dz,\end{aligned}$$

where

$$L_a(z)=\frac{1-\alpha(a^{-1}|z|)}{|z|^p}.$$

By the mean value theorem, there is a $C_a>0$ such that when $\varepsilon>0$ is sufficiently small,

$$|L_a(z-\varepsilon^{\gamma/d}x)-L_a(z-\varepsilon^{\gamma/d}y)|\le C_a\frac{\varepsilon^{\gamma/d}|x-y|}{|z|^p}1_{\{|z|\ge C_a^{-1}\}},\qquad x,y\in D.$$

Summarizing what we have,

$$\begin{aligned}&\mathbb{E}\exp\left\{\theta\varepsilon^{-p\gamma/d}\frac{|H_\varepsilon(x)-H_\varepsilon(y)|}{|x-y|}\right\}\\&\quad\le 2\exp\left\{\varepsilon^{-2/(d-p)}\int_{\{|x|\ge C_a^{-1}\}}\Psi\left(\frac{C_a\theta\varepsilon^{\gamma/d}}{|z|^p}\right)dz\right\}\\&\quad=2\exp\left\{\varepsilon^{\gamma/p-2/(d-p)}\int_{\{|x|\ge C_a^{-1}\varepsilon^{-\gamma/(dp)}\}}\Psi\left(\frac{C_a\theta}{|z|^p}\right)dz\right\}.\end{aligned}$$

Let $h>0$ satisfy that

$$h\ge\frac{3p-d-2}{d-p}\quad\text{or}\quad\frac{\gamma}{p}-\frac{2}{d-p}\ge 0.$$

Then for any $\theta>0$ the quantity

$$\sup_{\substack{x,y\in D\\x\ne y}}\mathbb{E}\exp\left\{\theta\varepsilon^{-p\gamma/d}\frac{|H_\varepsilon(x)-H_\varepsilon(y)|}{|x-y|}\right\}$$

is bounded uniformly for small $\varepsilon>0$. Thus ([8], Theorem D-6), for any $\theta>0$,

$$\lim_{\delta\to0^+}\limsup_{\varepsilon\to0^+}\mathbb{E}\exp\left\{\theta\varepsilon^{-p\gamma/d}\sup_{|x-y|\le\delta}|H_\varepsilon(x)-H_\varepsilon(y)|\right\}=1.\tag{3.21}$$

On the other hand, for any $x \in \mathbb{R}^d$ and $\theta > 0$,

$$\mathbb{E}\exp\{\pm\theta\varepsilon^{-p\gamma/d} H_\varepsilon(x)\} = \mathbb{E}\exp\{\pm\theta\varepsilon^{-p\gamma/d} H_\varepsilon(0)\} = \mathbb{E}\exp\bigl\{\pm\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\overline{V}^{(0)}(0)\bigr\},$$

where the last step follows from (3.20). By (3.17) and (3.18), therefore, for any $x \in D$

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\{-\theta\varepsilon^{-p\gamma/d} H_\varepsilon(x)\} = \int_{\mathbb{R}^d} \psi\biggl(\theta\frac{1-\alpha(a^{-1}|y|)}{|y|^p}\biggr)\,dy,$$

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\{\theta\varepsilon^{-p\gamma/d} |H_\varepsilon(x)|\} = \int_{\mathbb{R}^d} \Psi\biggl(\theta\frac{1-\alpha(a^{-1}|y|)}{|y|^p}\biggr)\,dy.$$

Combine them with (3.21). A standard argument of exponential approximation leads to

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\Bigl\{-\theta\varepsilon^{-p\gamma/d} \inf_{x\in D} H_\varepsilon(x)\Bigr\} = \int_{\mathbb{R}^d} \psi\biggl(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\biggr)\,dx, \tag{3.22}$$

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\Bigl\{\theta\varepsilon^{-p\gamma/d} \sup_{x\in D} |H_\varepsilon(x)|\Bigr\} = \int_{\mathbb{R}^d} \Psi\biggl(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\biggr)\,dx. \tag{3.23}$$

Recall that $D = (-b, b)^d$. Using (3.20),

$$-\inf_{x\in D} V_{a,\varepsilon}^{(0)}(x) = -\varepsilon^{-ph/d} \inf_{x\in\varepsilon^{-h/d}D} H_\varepsilon(x) \le \varepsilon^{-ph/d} \max_{z\in b\mathbb{Z}^d\cap\varepsilon^{-h/d}D} \Bigl\{-\inf_{x\in z+D} H_\varepsilon(x)\Bigr\}.$$

By the fact that the random variables

$$\inf_{x\in z+D} H_\varepsilon(x); \qquad z \in b\mathbb{Z}^d \cap \varepsilon^{-h/d} D,$$

are identically distributed,

$$\mathbb{E}\exp\Bigl\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))} \inf_{x\in D} \overline{V}_{a,\varepsilon}^{(0)}(x)\Bigr\} \le \#\{b\mathbb{Z}^d \cap \varepsilon^{-h/d} D\}\mathbb{E}\exp\Bigl\{-\theta\varepsilon^{-p\gamma/d} \inf_{x\in D} H_\varepsilon(x)\Bigr\}.$$

Consequently from (3.22),

$$\limsup_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E}\exp\Bigl\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))} \inf_{x\in D} \overline{V}_{a,\varepsilon}^{(0)}(x)\Bigr\} \le \int_{\mathbb{R}^d} \Psi\biggl(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\biggr)\,dx.$$

In view of (3.17), we have proved (3.14).

Assertion (3.15) follows from (3.18) and (3.23) in the same way. □

3.1. *Proof of Theorem* 3.1. Let $\theta > 0$ be fixed but arbitrary. By (3.15) and the inequality

$$\begin{aligned}\sup_{g\in\mathcal{G}_d(D)}|F_{a,\varepsilon}(g)| &\le \sup_{g\in\mathcal{G}_d(D)}\int_D\big|\overline{V}^{(0)}_{a,\varepsilon}(x)\big|g^2(x)\,dx\\ &\le\Big(\sup_{g\in\mathcal{G}_d(D)}\|g\|_{\mathcal{L}^2(D)}\Big)^2\sup_{x\in D}\big|\overline{V}^{(0)}_{a,\varepsilon}(x)\big|,\end{aligned}$$

we have

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\varepsilon^{2/(d-p)}\log\mathbb{E}\exp\Big\{\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\sup_{g\in\mathcal{G}_d(D)}\big|F^{(0)}_{a,\varepsilon}(g)\big|\Big\}\\ &\quad\le\int_{\mathbb{R}^d}\Psi\bigg(\Big(\sup_{g\in\mathcal{G}_d(D)}\|g\|_{\mathcal{L}^2(D)}\Big)^2\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^p}\bigg)\,dx.\end{aligned}$$

Consequently,

$$\lim_{a\to\infty}\limsup_{\varepsilon\to0^+}\varepsilon^{2/(d-p)}\log\mathbb{E}\exp\Big\{\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\sup_{g\in\mathcal{G}_d(D)}\big|F^{(0)}_{a,\varepsilon}(g)\big|\Big\}=0.$$

Therefore, (3.6) follows from a standard application of Chebyshev's inequality.

We now prove (3.8). For any $g\in\mathcal{G}_d(D)$,

$$\begin{aligned}&\mathbb{E}\exp\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\zeta_\varepsilon(g)\}\\ &\quad=\exp\bigg\{\varepsilon\int_{\mathbb{R}^d}\psi\bigg(\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\int_D\frac{1}{|y-x|^p}g^2(y)\,dy\bigg)\,dx\bigg\}.\end{aligned}$$

Given $\delta>0$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\psi\bigg(\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\int_D\frac{1}{|y-x|^p}g^2(y)\,dy\bigg)\,dx\\ &\quad\ge\int_{\{|x|\ge\delta\varepsilon^{-(2+d-p)/(d(d-p))}\}}\psi\bigg(\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\int_D\frac{1}{|y-x|^p}g^2(y)\,dy\bigg)\,dx\\ &\quad\ge\int_{\{|x|\ge\delta\varepsilon^{-(2+d-p)/(d(d-p))}\}}\psi\bigg(\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\frac{(1+o(1))}{|x|^p}\|g\|^2_{\mathcal{L}^2(D)}\bigg)\,dx\\ &\quad=\big(1+o(1)\big)\theta^{d/p}\|g\|^{2d/p}_{\mathcal{L}^2(D)}\varepsilon^{-(2+d-p)/(d-p)}\int_{\{|x|\ge(1+o(1))\delta\}}\psi\bigg(\frac{1}{|x|^p}\bigg)\,dx\\ &\qquad\qquad(\varepsilon\to0^+).\end{aligned}$$

Since δ can be arbitrarily small, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E} \exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))} \inf_{g\in\mathcal{G}_d(D)} \zeta_\varepsilon(g)\Big\} \\
&\quad \geq \theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)} \|g\|_{\mathcal{L}^2(D)}\Big)^{2d/p} \int_{\mathbb{R}^d} \psi\Big(\frac{1}{|x|^p}\Big)\,dx \qquad (3.24)\\
&\quad = \theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)} \|g\|_{\mathcal{L}^2(D)}\Big)^{2d/p} \frac{\omega_d p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big),
\end{aligned}$$

where the last step follows from (3.1).

On the other hand, for any $g \in \mathcal{G}_d(D)$ and $a > 0$,

$$\zeta_\varepsilon(g) = G_{a,\varepsilon}^{(0)}(g) + F_{a,\varepsilon}^{(0)}(g). \qquad (3.25)$$

Notice that

$$\begin{aligned}
G_{a,\varepsilon}^{(0)}(g) &\geq -\varepsilon \int_{\mathbb{R}^d}\Big[\int_D K_{a,\varepsilon}^{(0)}(y-x)g^2(y)\,dy\Big]dx \\
&= -\varepsilon\|g\|_{\mathcal{L}^2(D)}^2 \int_{\mathbb{R}^d} K_{a,\varepsilon}^{(0)}(x)\,dx \\
&\geq -Ca^{d-p}\varepsilon^{-(2-p)/d}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E} \exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))} \inf_{g\in\mathcal{G}_d(D)} \zeta_\varepsilon(g)\Big\} \\
&\quad \leq C\theta a^{d-p} \qquad (3.26)\\
&\qquad + \limsup_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E} \exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))} \inf_{g\in\mathcal{G}_d(D)} F_{a,\varepsilon}^{(0)}(g)\Big\}.
\end{aligned}$$

To deal with the right-hand side, notice that

$$F_{a,\varepsilon}^{(0)}(g) = \int_D \overline{V}_{a,\varepsilon}^{(0)}(x)g^2(x)\,dx \geq \|g\|_{\mathcal{L}^2(D)}^2 \inf_{x\in D} \overline{V}_{a,\varepsilon}^{(0)}(x).$$

Hence,

$$\inf_{g\in\mathcal{G}_d(D)} F_{a,\varepsilon}^{(0)}(g) = \inf_{g\in\mathcal{G}_d(D)} \int_D \overline{V}_{a,\varepsilon}^{(0)}(x)g^2(x)\,dx \geq \Big(\sup_{g\in\mathcal{G}_d(D)} \|g\|_{\mathcal{L}^2(D)}^2\Big) \inf_{x\in D} \overline{V}_{a,\varepsilon}^{(0)}(x)$$

when $\inf_{x\in D} \overline{V}_{a,\varepsilon}^{(0)}(x) \leq 0$, and

$$\inf_{g\in\mathcal{G}_d(D)} F_{a,\varepsilon}^{(0)}(g) = \inf_{g\in\mathcal{G}_d(D)} \int_D \overline{V}_{a,\varepsilon}^{(0)}(x)g^2(x)\,dx \geq \Big(\inf_{g\in\mathcal{G}_d(D)} \|g\|_{\mathcal{L}^2(D)}^2\Big) \inf_{x\in D} \overline{V}_{a,\varepsilon}^{(0)}(x)$$

when $\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)>0$. Thus,

$$
\begin{aligned}
&\mathbb{E}\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\inf_{g\in\mathcal{G}_d(D)}F^{(0)}_{a,\varepsilon}(g)\Big\}\\
&\quad\le\mathbb{E}\Big[\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\sup_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)\Big\};\\
&\qquad\qquad\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)\le 0\Big]\\
&\qquad+\mathbb{E}\Big[\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\inf_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)\Big\};\\
&\qquad\qquad\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)>0\Big]\\
&\quad\le\mathbb{E}\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\sup_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)\Big\}\\
&\qquad+\mathbb{E}\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\inf_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\inf_{x\in D}\overline{V}^{(0)}_{a,\varepsilon}(x)\Big\}.
\end{aligned}
$$

By (3.14) with θ being replaced by

$$
\theta\sup_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\quad\text{and}\quad\theta\inf_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)},
$$

respectively,

$$
\begin{aligned}
&\limsup_{\varepsilon\to0^+}\varepsilon^{2/(d-p)}\log\mathbb{E}\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\inf_{g\in\mathcal{G}_d(D)}F^{(0)}_{a,\varepsilon}(g)\Big\}\\
&\quad\le\int_{\mathbb{R}^d}\psi\bigg(\frac{\theta(1-\alpha(a^{-1}|x|))}{|x|^p}\sup_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\bigg)\,dx\\
&\quad\le\int_{\mathbb{R}^d}\psi\bigg(\frac{\theta}{|x|^p}\sup_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\bigg)\,dx\\
&\quad=\theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)}\|g\|^2_{\mathcal{L}^2(D)}\Big)^{2d/p}\int_{\mathbb{R}^d}\psi\bigg(\frac{1}{|x|^p}\bigg)\,dx\\
&\quad=\theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)}\|g\|_{\mathcal{L}^2(D)}\Big)^{2d/p}\frac{\omega_d p}{d-p}\Gamma\bigg(\frac{2p-d}{p}\bigg).
\end{aligned}
$$

Bringing this to (3.26),

$$
\begin{aligned}
&\limsup_{\varepsilon\to0^+}\varepsilon^{2/(d-p)}\log\mathbb{E}\exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))}\inf_{g\in\mathcal{G}_d(D)}\zeta_\varepsilon(g)\Big\}\\
&\quad\le C\theta a^{d-p}+\theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)}\|g\|_{\mathcal{L}^2(D)}\Big)^{2d/p}\frac{\omega_d p}{d-p}\Gamma\bigg(\frac{2p-d}{p}\bigg).
\end{aligned}
$$

Letting $a \to 0^+$ on the right-hand side leads to

$$\limsup_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{E} \exp\Big\{-\theta\varepsilon^{-p(2+d-p)/(d(d-p))} \inf_{g\in\mathcal{G}_d(D)} \zeta_\varepsilon(g)\Big\}$$
$$\le \theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)} \|g\|_{\mathcal{L}^2(D)}\Big)^{2d/p} \frac{\omega_d p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big). \tag{3.27}$$

The combination of (3.24) and (3.27) implies ([8], Theorem 1.2.4) that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{P}\Big\{\inf_{g\in\mathcal{G}_d(D)} \zeta_\varepsilon(g) \le -\gamma\varepsilon^{-(2-p)/d}\Big\}$$
$$= -\sup_{\theta>0}\Big\{\gamma\theta - \theta^{d/p}\Big(\sup_{g\in\mathcal{G}_d(D)} \|g\|_{\mathcal{L}^2(D)}\Big)^{2d/p} \frac{\omega_d p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big)\Big\}$$
$$= -I_D(\gamma).$$

It remains to prove (3.7). By (3.25), for any $\delta > 0$,

$$\mathbb{P}\Big\{\inf_{g\in\mathcal{G}_d(D)} \zeta_\varepsilon(g) \le -(\gamma+\delta)\varepsilon^{-(2-p)/d}\Big\}$$
$$\le \mathbb{P}\Big\{\inf_{g\in\mathcal{G}_d(D)} G_{a,\varepsilon}^{(0)}(g) \le -\gamma\varepsilon^{-(2-p)/d}\Big\}$$
$$+ \mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(D)} \big|F_{a,\varepsilon}^{(0)}(g)\big| \ge \delta\varepsilon^{-(2-p)/d}\Big\}.$$

Applying (3.8) on the left-hand side,

$$-I_D(\gamma+\delta) \le \max\Big\{\liminf_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{P}\Big\{\inf_{g\in\mathcal{G}_d(D)} G_{a,\varepsilon}^{(0)}(g) \le -\gamma\varepsilon^{-(2-p)/d}\Big\},$$
$$\limsup_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(D)} \big|F_{a,\varepsilon}^{(0)}(g)\big| \ge \delta\varepsilon^{-(2-p)/d}\Big\}\Big\}.$$

Let $a \to \infty$ on the right-hand side. By (3.6),

$$\liminf_{a\to\infty} \liminf_{\varepsilon\to 0^+} \varepsilon^{2/(d-p)} \log \mathbb{P}\Big\{\inf_{g\in\mathcal{G}_d(D)} G_{a,\varepsilon}^{(0)}(g) \le -\gamma\varepsilon^{-(2-p)/d}\Big\} \ge -I_D(\gamma+\delta).$$

Letting $\delta \to 0^+$ on the right-hand side leads to (3.7).

3.2. *Proof of Theorem* 3.2. Based on (3.16), assertion (3.11) follows from the same argument used in (3.6).

By the decomposition

$$G_{a,\varepsilon}^{(1)}(g) = \int_{\mathbb{R}^d}\Big[\int_D K_{a,\varepsilon}^{(1)}(y-x)g^2(y)\,dy\Big]\omega(\varepsilon\,dx)$$
$$- \varepsilon\int_{\mathbb{R}^d}\Big[\int_D K_{a,\varepsilon}^{(1)}(y-x)g^2(y)\,dy\Big]dx,$$

by the uniform [over $g \in \mathcal{G}_d(D)$] bound

$$\int_{\mathbb{R}^d}\left[\int_D K_{a,\varepsilon}^{(1)}(y-x)g^2(y)\,dy\right]dx = \left(\int_D g^2(y)\,dy\right)\left(\int_{\mathbb{R}^d} K_{a,\varepsilon}^{(1)}(x)\,dx\right)$$
$$= O\left(\left(\log\frac{1}{\varepsilon}\right)^{(d-p)/p}\right)$$

and by (3.11), all we need is to establish that

$$\lim_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(D)}\eta_{a,\varepsilon}(g)\ge\gamma\varepsilon^{-(2-p)/d}\Big\} = -\frac{2+d-p}{d\rho_D^*}\gamma, \tag{3.28}$$

where

$$\eta_{a,\varepsilon}(g) = \int_{\mathbb{R}^d}\left[\int_D K_{a,\varepsilon}^{(1)}(y-x)g^2(y)\,dy\right]\omega(\varepsilon\,dx).$$

Since $K_{a,\varepsilon}^{(1)}(y-x) = 0$ as $|y-x| > 3a(\log\varepsilon^{-1})^{1/p}$, x is limited to a ball with the center 0 and the radius $C(\log\varepsilon^{-1})^{1/p}$ when $y \in D$. Consequently,

$$\begin{aligned}\sup_{g\in\mathcal{G}_d(D)}\eta_{a,\varepsilon}(g) &\le \sup_{g\in\mathcal{G}_d(D)}\int_{\{|x|\le C(\log\varepsilon^{-1})^{1/p}\}}\left[\int_D\frac{g^2(y)}{|y-x|^p}\,dy\right]\omega(\varepsilon\,dx)\\ &\le \rho_D^*\omega\{|x|\le C\varepsilon^{1/d}(\log\varepsilon^{-1})^{1/p}\}\\ &= \rho_D^*\widetilde{Z}_\varepsilon,\end{aligned} \tag{3.29}$$

where $\widetilde{Z}_\varepsilon \equiv \omega\{|x|\le C\varepsilon^{1/d}(\log\varepsilon^{-1})^{1/p}\}$ is a Poisson random variable with

$$\mathbb{E}\widetilde{Z}_\varepsilon = \omega_d C^d\varepsilon(\log\varepsilon^{-1})^{d/p}.$$

For any $\theta > 0$

$$\mathbb{E}\exp\left\{\theta\left(\log\frac{1}{\varepsilon}\right)\widetilde{Z}_\varepsilon\right\} = \exp\{\omega_d C^d\varepsilon(\log\varepsilon^{-1})^{d/p}(e^{\theta\log\varepsilon^{-1}}-1)\}.$$

Consequently,

$$\lim_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{E}\exp\left\{\theta\left(\log\frac{1}{\varepsilon}\right)\widetilde{Z}_\varepsilon\right\} = 0, \qquad \theta < \frac{2+d-p}{d}.$$

A standard application of Chebyshev's inequality gives

$$\limsup_{\varepsilon\to 0^+}\frac{1}{l(\varepsilon)}\log\mathbb{P}\{\widetilde{Z}_\varepsilon\ge\gamma\varepsilon^{(2-p)/d}\} \le -\frac{2+d-p}{d}\gamma$$

for every $\gamma > 0$. Thus, the upper bound of (3.28) follows from (3.29).

On the other hand, let $x_0 \in \mathbb{R}^d$ be fixed but arbitrary, and write $\omega_{x_0}(\varepsilon\, dx) = \omega(\varepsilon(x_0 + dx))$. Given $\delta > 0$ and $\lambda > 1$, by variable shifting

$$\begin{aligned}
&\sup_{g\in\mathcal{G}_d(D)} \eta_{a,\varepsilon}(g)\\
&\quad = \sup_{g\in\mathcal{G}_d(D)} \int_{\mathbb{R}^d}\left[\int_D K_{a,\varepsilon}^{(1)}(y - x_0 - x) g^2(y)\, dy\right]\omega_{x_0}(\varepsilon\, dx)\\
&\quad \ge \sup_{g\in\mathcal{G}_d(D)} \int_{\{|x|\le\delta\}}\left[\int_D K_{a,\varepsilon}^{(1)}(y - x_0 - x) g^2(y)\, dy\right]\omega_{x_0}(\varepsilon\, dx)\\
&\quad \ge \left(\sup_{g\in\mathcal{G}_d(D)} \int_D \frac{1}{(|y-x_0|+\delta)^p}\alpha\big(a^{-1}(\log\varepsilon^{-1})^{-1/p}(|y-x_0|+\delta)\big)g^2(y)\, dy\right)\\
&\qquad \times \omega\{|x+x_0| \le \varepsilon^{1/d}\delta\}\\
&\quad \overset{d}{=} \left(\sup_{g\in\mathcal{G}_d(D)} \int_D \frac{g^2(y)}{(|y-x_0|+\delta)^p}\, dy\right)\omega\{|x| \le \varepsilon^{1/d}\delta\}
\end{aligned}$$

as ε is sufficiently small.

Write $Z_\varepsilon = \omega\{|x| \le \varepsilon^{1/d}\delta\}$ and $k(\varepsilon) = [\gamma\varepsilon^{-(2-p)/d}] + 1$.

$$\mathbb{P}\{Z_\varepsilon \ge \gamma\varepsilon^{-(2-p)/d}\} \ge \mathbb{P}\{Z_\varepsilon = k(\varepsilon)\} = e^{-\omega_d\varepsilon\delta^d}\frac{(\omega_d\varepsilon\delta^d)^{k(\varepsilon)}}{k(\varepsilon)!}.$$

By Stirling's formula, one can show that for any $\gamma > 0$,

$$\liminf_{\varepsilon\to\infty} \frac{1}{l(\varepsilon)} \log \mathbb{P}\{Z_\varepsilon \ge \gamma\varepsilon^{-(2-p)/d}\} \ge -\frac{2+d-p}{d}\gamma, \qquad \gamma > 0.$$

Replacing γ by

$$\gamma\left(\sup_{g\in\mathcal{G}_d(D)} \int_D \frac{g^2(y)}{(|y-x_0|+\delta)^p}\, dy\right)^{-1},$$

we have

$$\begin{aligned}
&\liminf_{\varepsilon\to\infty} \frac{1}{l(\varepsilon)} \log \mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(D)} \eta_{a,\varepsilon}(g) \ge \gamma\varepsilon^{-(2-p)/d}\Big\}\\
&\qquad \ge -\frac{2+d-p}{d}\left(\sup_{g\in\mathcal{G}_d(D)} \int_D \frac{g^2(y)}{(|y-x_0|+\delta)^p}\, dy\right)^{-1}\gamma.
\end{aligned}$$

Letting $\delta \to 0^+$ and taking $x_0 \in \mathbb{R}^d$ on the right-hand side lead to the lower bound of (3.28).

4. Bridging to the eigenvalue problem. Throughout this section, let $D \subset \mathbb{R}^d$ be a bounded open domain, and let

$$\mathcal{F}_d(D) = \left\{ g \in W^{1,2}(D); \int_D g^2(x)\,dx = 1 \right\}. \tag{4.1}$$

Given a measurable function $\xi(x)$ on $\mathbb{R}^d$, we introduce the notation

$$\lambda_\xi(D) = \sup_{g \in \mathcal{F}_d(D)} \left\{ \int_D \xi(x) g^2(x)\,dx - \frac{1}{2}\int_D |\nabla g(x)|^2\,dx \right\}.$$

Clearly, $\lambda_\xi(D) \le \lambda_\eta(D)$ whenever $\xi(x) \le \eta(x)$ ($x \in D$).

Write

$$\tau_D = \inf\{s \ge 0; B_s \notin D\}.$$

It is well known that by the Feynman–Kac formula,

$$\mathbb{E}_0\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right] \approx \exp\{t\lambda_\xi(D)\} \qquad (t \to \infty)$$

in some proper sense. For the applications to our setting, some more explicit bounds are needed. This is our objective in this section.

LEMMA 4.1. *The inequality*

$$\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right] dx \le |D| \exp\{t\lambda_\xi(D)\} \tag{4.2}$$

holds regardless whether $\lambda_\xi(D)$ is finite or infinite.

PROOF. The argument in the case when $\xi(x) \le N$ for some constant $N > 0$ is classic (see the treatment given e.g., in [8], Section 4.1): A standard argument through a spectral theory [the boundedness of $\xi(\cdot)$ guarantees the boundedness of the underlined linear operators in the argument] gives that for any $g \in W^{1,2}(D)$

$$\int_D g(x)\mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\right\} g(B_t); \tau_D \ge t\right] dx \le \|g\|^2_{\mathcal{L}^2(D)} \exp\{t\lambda_\xi(D)\}.$$

In particular, let $g_n \in W^{1,2}(D)$ be a monotonic sequence such that $0 \le g_n(x) \le 1$ and $g_n(x) \uparrow 1$ ($n \to \infty$) for every $x \in D$. Then

$$\int_D g_n(x)\mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\right\} g_n(B_t); \tau_D \ge t\right] dx \le |D| \exp\{t\lambda_\xi(D)\},$$
$$n = 1, 2, \ldots.$$

Letting $n \to \infty$ on the left-hand side, the desired bound follows from monotonic convergence.

To remove the boundedness assumption, we write $\xi_N(x) = \min\{\xi(x), N\}$. By what has been proved,

$$\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi_N(B_s)\,ds\right\}; \tau_D \ge t\right]dx \le |D|\exp\{t\lambda_{\xi_N}(D)\} \le |D|\exp\{t\lambda_\xi(D)\}.$$

The conclusion follows from monotonic convergence again as we let $N \to \infty$ on the left-hand side. $\square$

LEMMA 4.2. *For any* $\alpha, \beta > 1$ *satisfying* $\alpha^{-1} + \beta^{-1} = 1$ *and* $\lambda_{(\beta/\alpha)\xi}(D) < \infty$ [*in this case* $\lambda_{\alpha^{-1}\xi}(D) < \infty$] *and* $0 < \delta < t$

$$\begin{aligned}
&\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right]dx \\
&\qquad \ge (2\pi)^{\alpha d/2}\delta^{d/2}t^{\alpha d/(2\beta)}|D|^{-2\alpha/\beta} \\
&\qquad\qquad \times \exp\{-\delta(\alpha/\beta)\lambda_{(\beta/\alpha)\xi}(D)\}\exp\{\alpha(t+\delta)\lambda_{\alpha^{-1}\xi}(D)\}.
\end{aligned}\tag{4.3}$$

PROOF. We only need to show that

$$\begin{aligned}
&\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right]dx \\
&\qquad \ge (2\pi)^{\alpha d/2}\delta^{d/2}t^{\alpha d/(2\beta)}|D|^{-\alpha/\beta}\exp\{\alpha(t+\delta)\lambda_{\alpha^{-1}\xi}(D)\} \\
&\qquad\qquad \times \left\{\int_D \mathbb{E}_x\left[\exp\left\{\frac{\beta}{\alpha}\int_0^\delta \xi(B_s)\,ds\right\}; \tau_D \ge \delta\right]dx\right\}^{-\alpha/\beta}
\end{aligned}\tag{4.4}$$

as, by Lemma 4.1,

$$\int_D \mathbb{E}_x\left[\exp\left\{\frac{\beta}{\alpha}\int_0^\delta \xi(B_s)\,ds\right\}; \tau_D \ge \delta\right]dx \le |D|\exp\{-\delta(\alpha/\beta)\lambda_{(\beta/\alpha)\xi}(D)\}.$$

We first consider the case when $\xi(x)$ is Hölder continuous. By the Feynman–Kac representation,

$$u(t,x) = \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right]$$

solves the initial-boundary value problem

$$\begin{cases}
\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \xi(x)u(t,x), & (t,x) \in (0,t) \times D, \\
u(0,x) = 1, & x \in D, \\
u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial D.
\end{cases}$$

Let $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \cdots$ be the eigenvalues of the operator $(1/2)\Delta + \xi$ in $\mathcal{L}^2(D)$ with zero boundary condition and initial value 1 in D, and let $e_k \in \mathcal{L}^2(D)$ be an orthonormal basis corresponding to $\{\lambda_k\}$. By (2.31) in [17],

$$\mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}\delta_x(B_t); \tau_D \ge t\right] = \sum_{k=1}^\infty e^{t\lambda_k}e_k^2(x) \ge e^{t\lambda_1}e_1^2(x).$$

Noticing the fact that $\lambda_1 = \lambda_\xi(D)$ and integrating both sides we have

$$\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}\delta_x(B_t);\, \tau_D \ge t\right]dx \ge \exp\{t\lambda_\xi(D)\}.$$

Replace ξ by $\alpha^{-1}\xi$ and t by $t+\delta$. By Hölder's inequality,

$$\begin{aligned}
&\exp\{(t+\delta)\lambda_{\alpha^{-1}\xi}(D)\}\\
&\quad\le \int_D \mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t+\delta} \xi(B_s)\,ds\right\}\delta_x(B_{t+\delta});\, \tau_D \ge t+\delta\right]dx\\
&\quad\le \left\{\int_D \mathbb{E}_x\left[\exp\left\{(\beta/\alpha)\int_t^{t+\delta} \xi(B_s)\,ds\right\};\, \tau_D \ge t+\delta\right]dx\right\}^{1/\beta}\\
&\quad\quad\times \left\{\int_D \mathbb{E}_x\left[\exp\left\{\int_0^{t} \xi(B_s)\,ds\right\}\delta_x(B_{t+\delta});\, \tau_D \ge t+\delta\right]dx\right\}^{1/\alpha}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}\delta_x(B_{t+\delta});\, \tau_D \ge t+\delta\right]\\
&\quad\le \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}\delta_x(B_{t+\delta});\, \tau_D \ge t\right]\\
&\quad= \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}p_\delta(B_t - x);\, \tau_D \ge t\right],
\end{aligned}$$

where

$$p_\delta(y) = \frac{1}{(2\pi\delta)^{d/2}}\exp\left\{-\frac{|y|^2}{2\delta}\right\} \le \frac{1}{(2\pi\delta)^{d/2}}.$$

In addition,

$$\begin{aligned}
&\int_D \mathbb{E}_x\left[\exp\left\{(\beta/\alpha)\int_t^{t+\delta} \xi(B_s)\,ds\right\};\, \tau_D \ge t+\delta\right]dx\\
&\quad\le \int_D \mathbb{E}_x\left[\exp\left\{(\beta/\alpha)\int_t^{t+\delta} \xi(B_s)\,ds\right\};\, B_t \in D,\, \tau_D' \ge t+\delta\right]dx\\
&\quad= \int_D \left[\int_D p_t(y-x)\mathbb{E}_y\left(\exp\left\{(\beta/\alpha)\int_0^{\delta} \xi(B_s)\,ds\right\};\, \tau_D \ge \delta\right)dy\right]dx\\
&\quad\le \frac{1}{(2\pi t)^{d/2}}|D|\int_D \mathbb{E}_y\left[\exp\left\{(\beta/\alpha)\int_0^{\delta} \xi(B_s)\,ds\right\};\, \tau_D \ge \delta\right]dy,
\end{aligned}$$

where

$$\tau_D' = \inf\{s \ge t;\, B_s \notin D\}.$$

Summarizing our argument, we have established the bound (4.4).

We now move to the case when $\xi(x) \ge -N$ for some $N > 0$. For any Hölder-continuous $\eta(x)$ on D with $\eta(x) \le \xi(x)$ a.e. on D,

$$\begin{aligned}
&\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right] dx \\
&\quad \ge (2\pi)^{\alpha d/2}\delta^{d/2}t^{\alpha d/(2\beta)}|D|^{-\alpha/\beta}\exp\{\alpha(t+\delta)\lambda_{\alpha^{-1}\eta}(D)\} \\
&\qquad \times \left\{\int_D \mathbb{E}_x\left[\exp\left\{\frac{\beta}{\alpha}\int_0^\delta \eta(B_s)\,ds\right\}; \tau_D \ge \delta\right] dx\right\}^{-\alpha/\beta} \\
&\quad \ge (2\pi)^{\alpha d/2}\delta^{d/2}t^{\alpha d/(2\beta)}|D|^{-\alpha/\beta}\exp\{\alpha(t+\delta)\lambda_{\alpha^{-1}\eta}(D)\} \\
&\qquad \times \left\{\int_D \mathbb{E}_x\left[\exp\left\{\frac{\beta}{\alpha}\int_0^\delta \xi(B_s)\,ds\right\}; \tau_D \ge \delta\right] dx\right\}^{-\alpha/\beta}.
\end{aligned}$$

Let

$$\mathcal{H}_\xi = \{\eta(\cdot); \eta(x) \text{ is Hölder continuous on } D \text{ and } \eta(x) \le \xi(x) \text{ a.e. on } D\}.$$

Since $\xi(\cdot) \ge -N$, $\mathcal{H}_\xi \ne \phi$. Further, by standard approximation theory, $\mathcal{H}_\xi$ is rich enough to approximate ξ. More precisely, the desired bound follows from

$$\begin{aligned}
\sup_{\eta\in\mathcal{H}_\xi}\lambda_{\alpha^{-1}\eta}(D) &= \sup_{g\in\mathcal{F}_d(D)}\left\{\alpha^{-1}\sup_{\eta\in\mathcal{H}_\xi}\int_D \eta(x)g^2(x)\,dx - \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right\} \\
&= \lambda_{\alpha^{-1}\xi}(D).
\end{aligned}$$

To remove the boundedness assumption, we write $\xi_N(x) = \xi(x) \vee (-N)$. We have

$$\begin{aligned}
&\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t \xi_N(B_s)\,ds\right\}; \tau_D \ge t\right] dx \\
&\quad \ge (2\pi)^{\alpha d/2}\delta^{d/2}t^{\alpha d/(2\beta)}|D|^{-\alpha/\beta}\exp\{\alpha(t+\delta)\lambda_{\alpha^{-1}\xi_N}(D)\} \\
&\qquad \times \left\{\int_D \mathbb{E}_x\left[\exp\left\{\frac{\beta}{\alpha}\int_0^\delta \xi_N(B_s)\,ds\right\}; \tau_D \ge \delta\right] dx\right\}^{-\alpha/\beta}.
\end{aligned}$$

Noticing $\lambda_{\alpha^{-1}\xi_N}(D) \ge \lambda_{\alpha^{-1}\xi}(D)$ and letting $N \to \infty$, the monotonic convergence theorem leads to (4.4). □

LEMMA 4.3. *Let* $0 < \delta < t$, *and assume* $0 \in D$.

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \ge t\right] \\
&\quad \le \left(\mathbb{E}_0 \exp\left\{\beta\int_0^\delta \xi(B_s)\,ds\right\}\right)^{1/\beta} \qquad (4.5)\\
&\qquad \times \left\{\frac{1}{(2\pi\delta)^{d/2}}\int_D \mathbb{E}_x\left[\exp\left\{\alpha\int_0^{t-\delta} \xi(B_s)\,ds\right\}; \tau_D \ge t-\delta\right] dx\right\}^{1/\alpha}.
\end{aligned}$$

On the other hand,

$$\mathbb{E}_0 \exp\left\{\int_0^t \xi(B_s)\,ds\right\}$$
$$\geq \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^\delta \xi(B_s)\,ds\right\}\right)^{-\alpha/\beta} \tag{4.6}$$
$$\times \left\{\int_D p_\delta(x)\mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t-\delta} \xi(B_s)\,ds\right\}; \tau_D \geq t-\delta\right]dx\right\}^\alpha,$$

where $p_\delta(x)$ *is the density of* B_δ.

PROOF. By Hölder's inequality,

$$\mathbb{E}_0\left[\exp\left\{\int_0^t \xi(B_s)\,ds\right\}; \tau_D \geq t\right]$$
$$\leq \left(\mathbb{E}_0 \exp\left\{\beta\int_0^\delta \xi(B_s)\,ds\right\}\right)^{1/\beta}\left\{\mathbb{E}_0\left[\exp\left\{\alpha\int_\delta^t \xi(B_s)\,ds\right\}; \tau_D \geq t\right]\right\}^{1/\alpha}.$$

Write $\tau_D' = \inf\{s \geq \delta; B_s \notin D\}$. Claim (4.5) follows from the following procedure via Markov property:

$$\mathbb{E}_0\left[\exp\left\{\alpha\int_\delta^t \xi(B_s)\,ds\right\}; \tau_D \geq t\right]$$
$$\leq \mathbb{E}_0\left[\exp\left\{\alpha\int_\delta^t \xi(B_s)\,ds\right\}; B_\delta \in D, \tau_D' \geq t\right]$$
$$= \int_D p_\delta(x)\mathbb{E}_x\left[\exp\left\{\alpha\int_0^{t-\delta} \xi(B_s)\,ds\right\}; \tau_D \geq t-\delta\right]dx$$
$$\leq \frac{1}{(2\pi\delta)^{d/2}}\int_D \mathbb{E}_x\left[\exp\left\{\alpha\int_0^{t-\delta} \xi(B_s)\,ds\right\}; \tau_D \geq t-\delta\right]dx.$$

On the other hand,

$$\mathbb{E}_0\left[\exp\left\{\alpha^{-1}\int_\delta^t \xi(B_s)\,ds\right\}; B_\delta \in D, \tau_D' \geq t\right]$$
$$\leq \mathbb{E}_0\left[\exp\left\{-\alpha^{-1}\int_0^\delta \xi(B_s)\,ds\right\}\exp\left\{\alpha^{-1}\int_0^t \xi(B_s)\,ds\right\}\right]$$
$$\leq \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^\delta \xi(B_s)\,ds\right\}\right)^{1/\beta}\left\{\mathbb{E}_0 \exp\left\{\int_0^t \xi(B_s)\,ds\right\}\right\}^{1/\alpha}.$$

Thus, (4.6) follows from Markov property which claims that

$$\mathbb{E}_0\left[\exp\left\{\alpha^{-1}\int_\delta^t \xi(B_s)\,ds\right\}; B_\delta \in D, \tau_D' \geq t\right]$$
$$= \int_D p_\delta(x)\mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t-\delta} \xi(B_s)\,ds\right\}; \tau_D \geq t-\delta\right]dx. \qquad \square$$

5. Upper bounds. In this section we establish the upper bounds for Theorems 2.1 and 2.2. More precisely, we prove that

$$\begin{aligned}&\limsup_{t\to\infty} t^{-1}(\log t)^{-(d-p)/d}\\ &\qquad\times \log \mathbb{E}_0 \exp\biggl\{-\theta\int_0^t \overline{V}(B_s)\,ds\biggr\} \le \Lambda_0(\theta) \qquad \text{a.s.-}\mathbb{P}\end{aligned} \tag{5.1}$$

when $d/2 < p < d$, and

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t}\biggl(\frac{\log\log t}{\log t}\biggr)^{2/(2-p)}\\ &\qquad\times \log \mathbb{E}_0 \exp\biggl\{\theta\int_0^t \overline{V}(B_s)\,ds\biggr\} \le \Lambda_1(\theta) \qquad \text{a.s.-}\mathbb{P}\end{aligned} \tag{5.2}$$

when $d/2 < p < \min\{2, d\}$, where

$$\Lambda_0(\theta) = \frac{\theta d^2}{d-p}\biggl(\frac{\omega_d}{d}\Gamma\biggl(\frac{2p-d}{p}\biggr)\biggr)^{p/d}, \tag{5.3}$$

$$\Lambda_1(\theta) = \frac{1}{2}p^{p/(2-p)}(2-p)^{(4-p)/(2-p)}\biggl(\frac{d\theta\,\sigma(d,p)}{2+d-p}\biggr)^{2/(2-p)}. \tag{5.4}$$

The following notation will be used in this and the next sections. For any $R > 0$, $Q_R = (-R, R)^d$.

$$h_t = \begin{cases} (\log t)^{(d-p)/(2d)}, & \text{for the proof of (5.1)},\\ \biggl(\dfrac{\log t}{\log\log t}\biggr)^{1/(2-p)}, & \text{for the proof of (5.2)}.\end{cases} \tag{5.5}$$

Write $R_k = R_k(t) = (Mth_t)^k$ $(k = 1, 2, \dots)$ where the constant $M > 0$ is fixed but sufficiently large. Write $\xi(x) = -\overline{V}(x)$ in the proof of (5.1) and $\xi(x) = \overline{V}(x)$ in the proof of (5.2).

Finally we recall that for any open domain $D \subset \mathbb{R}^d$ containing 0,

$$\tau_D = \inf\{s \ge 0;\ B_s \notin D\}.$$

Consider the decomposition

$$\begin{aligned}&\mathbb{E}_0 \exp\biggl\{\theta\int_0^t \xi(B_s)\,ds\biggr\}\\ &\quad= \mathbb{E}_0\biggl[\exp\biggl\{\theta\int_0^t \xi(B_s)\,ds\biggr\};\ \tau_{Q_{R_1}} \ge t\biggr]\\ &\qquad+ \sum_{k=1}^{\infty}\mathbb{E}_0\biggl[\exp\biggl\{\theta\int_0^t \xi(B_s)\,ds\biggr\};\ \tau_{Q_{R_k}} < t \le \tau_{Q_{R_k}}\biggr]\\ &\quad\le \mathbb{E}_0\biggl[\exp\biggl\{\theta\int_0^t \xi(B_s)\,ds\biggr\};\ \tau_{Q_{R_1}} \ge t\biggr]\end{aligned} \tag{5.6}$$

$$+\sum_{k=1}^{\infty}(\mathbb{P}\{\tau_{Q_{R_k}}<t\})^{1/2}$$
$$\times\left\{\mathbb{E}_0\left[\exp\left\{2\theta\int_0^t\xi(B_s)\,ds\right\};\tau_{Q_{R_{k+1}}}\geq t\right]\right\}^{1/2}.$$

The well-known result on the Gaussian tail gives that

$$(\mathbb{P}\{\tau_{Q_{R_k}}<t\})^{1/2}\leq\exp\{-cR_k^2/t\}=\exp\{-cM^2t^{2k-1}h_t^{2k}\}.$$

Let $\alpha,\beta>1$ satisfy $\alpha^{-1}+\beta^{-1}=1$ with α close to 1. By (4.5) (with $\delta=1$) and Lemma 4.1,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t\xi(B_s)\,ds\right\};\tau_{Q_{R_1}}\geq t\right]\\
&\quad\leq\frac{1}{(2\pi)^{d/\alpha}}\left(\mathbb{E}_0\exp\left\{\theta\beta\int_0^1\xi_{R_1}(B_s)\,ds\right\}\right)^{1/\beta}\\
&\qquad\times\left\{\int_{Q_{R_1}}dx\,\mathbb{E}_x\left[\exp\left\{\theta\alpha\int_0^1\xi(B_s)\,ds\right\};\tau_{Q_{R_1}}\geq t-1\right]\right\}^{1/\alpha}\\
&\quad\leq\left(\frac{R_1}{\pi}\right)^{d/\alpha}\left(\mathbb{E}_0\exp\left\{\theta\beta\int_0^1\xi(B_s)\,ds\right\}\right)^{1/\beta}\exp\{(t-1)\lambda_{\theta\alpha\xi}(Q_{R_1})\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{2\theta\int_0^t\xi(B_s)\,ds\right\};\tau_{\widetilde{Q}_{R_{k+1}}}\geq t\right]\\
&\quad\leq\left(\frac{R_{k+1}}{\pi}\right)^{d/\alpha}\left(\mathbb{E}_0\exp\left\{2\theta\beta\int_0^1\xi(B_s)\,ds\right\}\right)^{1/\beta}\exp\{(t-1)\lambda_{2\theta\alpha\xi}(Q_{R_{k+1}})\}.
\end{aligned}$$

Summarizing our estimates since (5.6),

$$\begin{aligned}
&\mathbb{E}_0\exp\left\{\theta\int_0^t\xi(B_s)\,ds\right\}\\
&\quad\leq\left(\frac{R_1}{\pi}\right)^{d/\alpha}\left(\mathbb{E}_0\exp\left\{\theta\beta\int_0^1\xi(B_s)\,ds\right\}\right)^{1/\beta}\exp\{t\lambda_{\alpha\theta\xi}(Q_{R_1})\}\\
&\qquad+\left(\mathbb{E}_0\exp\left\{2\theta\beta\int_0^1\xi(B_s)\,ds\right\}\right)^{1/2\beta}\\
&\qquad\quad\times\sum_{k=1}^{\infty}\left(\frac{R_{k+1}}{\pi}\right)^{d/2\alpha}\exp\{-cM^2t^{2k-1}h_t^{2k}\}\exp\left\{\frac{t}{2}\lambda_{2\alpha\theta\xi}(Q_{R_{k+1}})\right\}.
\end{aligned}\tag{5.7}$$

To prove (5.1) and (5.2), therefore, all we need is to show that for any $\theta > 0$,

$$\lim_{t\to\infty} h_t^{-2}\lambda_{\theta\xi}(Q_t) \le \Lambda(\theta) \equiv \begin{cases} \Lambda_0(\theta), & \text{for the proof of (5.1),} \\ \Lambda_1(\theta), & \text{for the proof of (5.2).} \end{cases} \tag{5.8}$$

a.s.-$\mathbb{P}$.

Indeed, we apply (5.8) to the first term on the right-hand side of (5.7) (with t being replaced by $R_1 = Mth_t$ and θ being replaced by $\alpha\theta$). Notice that α can be arbitrarily close to 1. This term alone does not exceed the limit set in (5.1) and (5.2) if we let $\alpha \to 1^+$ after the limit for t. To control the infinite series on the right-hand side of (5.7), we apply (5.8) to each term with t being replaced by $R_{k+1} = (Mth_t)^{k+1}$ and with θ being replaced by $2\alpha\theta$. In this way, the series is dominated by

$$\sum_{k=1}^{\infty}\left(\frac{R_{k+1}}{\pi}\right)^{d/\alpha} \exp\{-c' t^{2k-2} h_t^{2k}\} = O(1) \qquad \text{a.s.-}\mathbb{P}\ (t\to\infty),$$

where $c' > 0$ is a constant. Here we point out that to control the first term of the series in (5.7), $M > 0$ is required to be sufficiently large.

Let $\delta > 0$ be a small number, and write

$$\tilde{h}_t = h_t\sqrt{\frac{u}{1+\delta}}.$$

Define

$$\xi_t(x) = \pm\theta\tilde{h}_t^{p-2}\int_{\mathbb{R}^d}\frac{1}{|y-x|^p}[\omega(\tilde{h}_t^{-d}dx) - \tilde{h}_t^{-d}\,dx],$$

where "$-$" corresponds to the proof of (5.1) and "$+$" corresponds to the proof of (5.2).

Under the substitution

$$g(x) \mapsto \tilde{h}_t^{d/2} g(x\tilde{h}_t),$$

we have that

$$\lambda_{\theta\xi}(Q_t) = \tilde{h}_t^2 \sup_{g\in\mathcal{F}_d(Q_{t\tilde{h}_t})}\left\{\int_{Q_{t\tilde{h}_t}}\xi_t(x)g^2(x)\,dx - \frac{1}{2}\int_{Q_{t\tilde{h}_t}}|\nabla g(x)|^2\,dx\right\}.$$

Let $r \ge 2$ be large but fixed. By Proposition 1 in [16], or by Lemma 4.6 in [17], there is a nonnegative and continuous function $\Phi(x)$ on $\mathbb{R}^d$ whose support is contained in the 1-neighborhood of the grid $2r\mathbb{Z}^d$, such that

$$\lambda_{\xi_t-\Phi^y}(Q_{t\tilde{h}_t}) \le \max_{z\in 2r\mathbb{Z}^d\cap Q_{2t\tilde{h}_t+2r}} \lambda_{\xi_t}(z+Q_{r+1}), \qquad y\in Q_r,$$

where $\Phi^y(x) = \Phi(x+y)$. In addition, $\Phi(x)$ is periodic with period $2r$

$$\Phi(x+2rz) = \Phi(x); \qquad x\in\mathbb{R}^d, z\in\mathbb{Z}^d,$$

and there is a constant $K>0$ independent of r and t such that

$$\int_{Q_r}\Phi(x)\,dx\le\frac{K}{r}.$$

By periodicity

$$\begin{aligned}
&\sup_{g\in\mathcal{F}_d(Q_{t\tilde{h}_t})}\left\{\int_{Q_{t\tilde{h}_t}}\xi_t(x)g^2(x)\,dx-\frac{1}{2}\int_{Q_{t\tilde{h}_t}}|\nabla g(x)|^2\,dx\right\}\\
&\quad\le\frac{K}{r(2r)^d}+\sup_{g\in\mathcal{F}_d(Q_{t\tilde{h}_t})}\left\{\int_{Q_{t\tilde{h}_t}}\left(\xi_t(x)-\frac{1}{(2r)^d}\int_{Q_r}\Phi^y(x)dy\right)g^2(x)\,dx\right.\\
&\qquad\qquad\left.-\frac{1}{2}\int_{Q_{t\tilde{h}_t}}|\nabla g(x)|^2\,dx\right\}\\
&\quad\le\frac{K}{r(2r)^d}+\frac{1}{(2r)^d}\int_{Q_r}\sup_{g\in\mathcal{F}_d(Q_{t\tilde{h}_t})}\left\{\int_{Q_{t\tilde{h}_t}}\big(\xi_t(x)-\Phi^y(x)\big)g^2(x)\,dx\right.\\
&\qquad\qquad\left.-\frac{1}{2}\int_{Q_{Q_{t\tilde{h}_t}}}|\nabla g(x)|^2\,dx\right\}dy\\
&\quad=\frac{K}{r(2r)^d}+\frac{1}{(2r)^d}\int_{Q_r}\lambda_{\xi_t-\Phi^y}(Q_{t\tilde{h}_t})\,dy\\
&\quad\le\frac{K}{2^dr^{d+1}}+\max_{z\in 2r\mathbb{Z}^d\cap Q_{2t\tilde{h}_t+2r}}\lambda_{\xi_t}(z+Q_{r+1}).
\end{aligned}$$

Summarizing our estimates

$$\lambda_{\theta\xi}(Q_t)\le\frac{uh_t^2}{1+\delta}\left\{\frac{K}{2^dr^{d+1}}+\max_{z\in 2r\mathbb{Z}^d\cap Q_{2t\tilde{h}_t+2r}}\lambda_{\xi_t}(z+Q_{r+1})\right\}.$$

Take $r>0$ sufficiently large so that the first term on the right-hand side is less than $\frac{\delta u}{1+\delta}h_t^2$. We have that

$$\mathbb{P}\{\lambda_{\theta\xi}(Q_t)\ge uh_t^2\}\le\mathbb{P}\left\{\max_{z\in 2r\mathbb{Z}^d\cap Q_{2t\tilde{h}_t+2r}}\lambda_{\xi_t}(z+Q_{r+1})>1\right\}.\tag{5.9}$$

By shifting invariance of the Poisson field, the random variables

$$\lambda_{\xi_t}(z+Q_{r+1});\qquad z\in 2r\mathbb{Z}^d\cap Q_{2t\tilde{h}_t+2r},$$

are identically distributed. Consequently, there is $C>0$

$$\begin{aligned}
&\mathbb{P}\left\{\max_{z\in 2r\mathbb{Z}^d\cap Q_{2t\tilde{h}_t+2r}}\lambda_{\xi_t}(z+Q_{r+1})>1\right\}\\
&\quad\le C(th_t)^d\,\mathbb{P}\{\lambda_{\xi_t}(Q_{r+1})>1\}\qquad(5.10)\\
&\quad=C(th_t)^d\,\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_{r+1})}\int_{Q_{r+1}}\xi_t(x)g^2(x)\,dx>1\right\},
\end{aligned}$$

where $\mathcal{G}_d(Q_{r+1})$ is defined in (3.2) and the last step follows from Lemma A.2.

We now reach the point of applying Theorems 3.1 and 3.2. In connection with (5.1),

$$\sup_{g\in\mathcal{G}_d(Q_{r+1})}\int_{Q_{r+1}}\xi_t(x)g^2(x)\,dx$$
$$=-\theta\tilde{h}_t^{p-2}\inf_{g\in\mathcal{G}_d(Q_{r+1})}\int_{\mathbb{R}^d}\biggl[\int_{Q_{r+1}}\frac{g^2(y)}{|y-x|^p}\,dy\biggr][\omega(\tilde{h}_t^{-d}\,dx)-\tilde{h}_t^{-d}\,dx].$$

Taking $\varepsilon=\tilde{h}_t^{-d}$ and $\gamma=\theta^{-1}$ in (3.8) leads to

$$\lim_{t\to\infty}\frac{1}{\log t}\log\mathbb{P}\biggl\{\sup_{g\in\mathcal{G}_d(Q_{r+1})}\int_{Q_{r+1}}\xi_t(x)g^2(x)\,dx>1\biggr\}$$
$$=-\biggl(\frac{u}{1+\delta}\biggr)^{d/(d-p)}I_{Q_{r+1}}(\theta^{-1}) \tag{5.11}$$
$$\le-\biggl(\frac{u(d-p)}{\theta d(1+\delta)}\biggr)^{d/(d-p)}\biggl(\omega_d\Gamma\biggl(\frac{2p-d}{p}\biggr)\biggr)^{-p/(d-p)},$$

where the rate function $I_{Q_{r+1}}(\cdot)$ is defined in (3.9), and the last step follows from the obvious fact that

$$\sup_{g\in\mathcal{G}_d(Q_{r+1})}\|g\|_{\mathcal{L}^2(Q_{r+1})}\le 1.$$

Take $u=(1+2\delta)\Lambda(\theta)$. By (5.9), (5.10) and (5.11), there is a $\nu>0$ such that

$$\mathbb{P}\{\lambda_{\theta\xi}(Q_t)\ge(1+2\delta)\Lambda(\theta)h_t^2\}\le C(th_t)^d\exp\{(d+\nu)\log t\}=C\frac{h_t^d}{t^\nu} \tag{5.12}$$

for sufficiently large t.

We now establish (5.12) for the proof of (5.2). In this case

$$\sup_{g\in\mathcal{G}_d(Q_{r+1})}\int_{Q_{r+1}}\xi_t(x)g^2(x)\,dx$$
$$=\theta\tilde{h}_t^{p-2}\sup_{g\in\mathcal{G}_d(Q_{r+1})}\int_{\mathbb{R}^d}\biggl[\int_{Q_{r+1}}\frac{g^2(y)}{|y-x|^p}\,dy\biggr][\omega(\tilde{h}_t^{-d}dx)-\tilde{h}_t^{-d}\,dx].$$

Taking $\varepsilon=\tilde{h}_t^{-d}$ and $\gamma=\theta^{-1}$ in (3.13),

$$\lim_{t\to\infty}\frac{1}{\log t}\log\mathbb{P}\biggl\{\sup_{g\in\mathcal{G}_d(Q_{r+1})}\int_{Q_{r+1}}\xi_t(x)g^2(x)\,dx>1\biggr\}$$
$$=-\biggl(\frac{u}{1+\delta}\biggr)^{(2-p)/2}\frac{2+d-p}{\theta(2-p)\rho^*_{Q_{r+1}}},$$

where $\rho^*_{Q_{r+1}}$ is defined as the second variation in (3.10) with $D=Q_{r+1}$.

Write

$$\mathcal{G}_d = \mathcal{G}_d(\mathbb{R}^d) = \left\{ g \in W^{1,2}(\mathbb{R}^d); \|g\|_2 + \frac{1}{2}\|\nabla g\|_2^2 = 1 \right\}, \tag{5.13}$$

$$\begin{aligned} \rho(d,p) &= \sup_{g\in\mathcal{G}_d} \int_{\mathbb{R}^d} \frac{g^2(x)}{|y|^p}\,dy \quad \text{and} \\ \rho^*(d,p) &= \sup_{g\in\mathcal{G}_d}\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g^2(x)}{|y-x|^p}\,dy. \end{aligned} \tag{5.14}$$

Clearly, $\rho^*_{Q_{r+1}} \le \rho^*(d,p)$. By (A.9), $\rho^*(d,p) = \rho(d,p)$.

By (A.7) in Lemma A.2, therefore,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\left\{ \sup_{g\in\mathcal{G}_d(Q_{r+1})} \int_{Q_{r+1}} \xi_t(x) g^2(x)\,dx > 1 \right\} \\ &\quad \le -p^{-p/2}(2-p)^{-(4-p)/2}\left(\frac{2u}{1+\delta}\right)^{(2-p)/2} \frac{2+d-p}{\theta\sigma(d,p)}. \end{aligned} \tag{5.15}$$

Again, (5.12) [in the context of (5.2)] follows forms (5.9), (5.10) and (5.15).

For any $\gamma > 1$, (5.12) implies that

$$\sum_k \mathbb{P}\{\lambda_{\theta\xi}(Q_{\gamma^k}) \ge (\Lambda(\theta)+\delta) h_{\gamma^k}^2\} < \infty.$$

By the Borel–Cantelli lemma,

$$\limsup_{k\to\infty} h_{\gamma^k}^{-2} \lambda_{\theta\xi}(Q_{\gamma^k}) \le (1+2\delta)\Lambda(\theta) \qquad \text{a.s.}$$

Since $\lambda_{\theta\xi}(Q_t)$ is monotonic in t and $\delta > 0$ can be arbitrarily small, we have proved (5.8).

6. Lower bounds.

In this section we prove that

$$\liminf_{t\to\infty} t^{-1}(\log t)^{-(d-p)/d} \log \mathbb{E}_0 \exp\left\{ -\theta \int_0^t \overline{V}(B_s)\,ds \right\} \ge \Lambda_0(\theta) \qquad \text{a.s.-}\mathbb{P} \tag{6.1}$$

when $d/2 < p < d$ and

$$\liminf_{t\to\infty} \frac{1}{t}\left(\frac{\log\log t}{\log t}\right)^{2/(2-p)} \log \mathbb{E}_0 \exp\left\{ \theta \int_0^t \overline{V}(B_s)\,ds \right\} \ge \Lambda_1(\theta) \qquad \text{a.s.-}\mathbb{P} \tag{6.2}$$

when $d/2 < p < \min\{2,d\}$; where $\Lambda_0(\theta)$ and $\Lambda_1(\theta)$ are given in (5.3) and (5.4), respectively.

Let h_t be defined in (5.5), and write $\xi(x) = -\overline{V}(x)$ in connection with the proof of (6.1) and $\xi(x) = \overline{V}(x)$ in connection with the proof of (6.2). Let $0 < q < 1$ be fixed but close to 1. Let $\alpha, \beta > 1$ satisfy $\alpha^{-1} + \beta^{-1} = 1$ with α being close to 1. By (4.6) in Lemma 4.3,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \xi(B_s)\,ds\right\} \\
&\quad \ge \left(\mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^q} \xi(B_s)\,ds\right\}\right)^{-\alpha/\beta} \\
&\qquad \times \left\{\int_{Q_{t^q}} p_{t^q}(x)\mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t-t^q} \xi(B_s)\right\}; \tau_{Q_{t^q}} \ge t - t^q\right]dx\right\}^{\alpha} \\
&\quad \ge \frac{1}{(2\pi t^q)^{\alpha d/2}} e^{-ct^q}\left(\mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^q} \xi(B_s)\,ds\right\}\right)^{-\alpha/\beta} \qquad (6.3)\\
&\qquad \times \left\{\int_{Q_{t^q}} \mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t-t^q} \xi(B_s)\right\}; \tau_{Q_{t^q}} \ge t - t^q\right]dx\right\}^{\alpha} \\
&\quad \ge e^{-c_1 t^q}\left(\mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^q} \xi(B_s)\,ds\right\}\right)^{-\alpha/\beta} \\
&\qquad \times \exp\{-(\alpha^2/\beta)t^q \lambda_{(\beta/\alpha^2)\theta\xi}(Q_{t^q}) + \alpha^2 t \lambda_{\alpha^{-2}\theta\xi}(Q_{t^q})\}
\end{aligned}$$

for large t, where the last step follows from Lemma 4.2 (with $\delta = t^q$ and t being replaced by $t - t^q$), and the positive constant c_1 is made to be larger than c for absorbing all bounded-by-polynomial quantities including those appearing on the right-hand side of (4.3).

By (5.1), (5.2) and (5.8)

$$\log \mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^q} \xi(B_s)\,ds\right\} = o(t) \quad \text{and}$$

$$\lambda_{(\beta/\alpha^2)\theta\xi}(Q_{t^q}) = O(h_t^2) \qquad \text{a.s.}$$

as $t \to \infty$. Therefore, all we need is to show that

$$\liminf_{t\to\infty} h_t^{-2} \lambda_{\theta\xi}(Q_t) \ge \Lambda(\theta) \qquad \text{a.s.} \tag{6.4}$$

for every $\theta > 0$, where $\Lambda(\theta)$ is given in (5.8). Indeed, applying (6.4) to (6.3) with θ being replaced by $\alpha^{-2}\theta$ leads to

$$\liminf_{t\to\infty} t^{-1} h_{t^q}^{-2} \log \mathbb{E}_0 \exp\left\{\theta\int_0^t \xi(B_s)\,ds\right\} \ge \alpha^2 \Lambda(\alpha^{-2}\theta) \qquad \text{a.s.}$$

Letting $\alpha \to 1^+$, the right-hand side tends to $\Lambda(\theta)$. In addition, $h_{t^q} = q^{(d-p)/(2d)} h_t$ when applied to (6.1) and $h_{t^q} \sim q^{1/(2-p)} h_t$ when applied to (6.2). Therefore, with

probability 1,

$$\liminf_{t\to\infty} t^{-1}h_t^{-2}\log\mathbb{E}_0\exp\left\{\theta\int_0^t\xi(B_s)\,ds\right\}$$
$$\geq\begin{cases} q^{(d-p)/d}\Lambda_0(\theta), & \text{when applied to (6.1)},\\ q^{2/(2-p)}\Lambda_1(\theta), & \text{when applied to (6.2)}.\end{cases}$$

Letting $q\to 1^-$ on the right-hand side leads to (6.1) and (6.2).

We now prove (6.4). Let $u>0$ be fixed but arbitrary. Write $\hat{h}_t=\sqrt{u}h_t$ and

$$\eta_t(x)=\pm\theta\hat{h}_t^{p-2}\int_{\mathbb{R}^d}\frac{1}{|y-x|^p}[\omega(\hat{h}_t^{-d}\,dy)-\hat{h}_t^{-d}\,dx],$$

where "$-$" is for the proof of (6.1) and "$+$" is for the proof of (6.2). Under the substitution $g(x)\mapsto\hat{h}_t^{d/2}g(h_tx)$,

$$\lambda_{\theta\xi}(Q_t)=\hat{h}_t^2\lambda_{\eta_t}(Q_{t\hat{h}_t}).$$

Consequently,

$$\begin{aligned}\mathbb{P}\{\lambda_{\theta\xi}(Q_t)\leq uh_t^2\}&=\mathbb{P}\{\lambda_{\eta_t}(Q_{t\hat{h}_t})\leq 1\}\\ &=\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_{t\hat{h}_t})}\int_{\mathbb{R}^d}\eta_t(x)g^2(x)\,dx\leq 1\right\},\end{aligned}\tag{6.5}$$

where the last step follows from Lemma A.2.

Let $s>\frac{2+d-p}{d-p}$ and $r>0$ be fixed. When t is large, $z+Q_r\subset Q_{t\hat{h}_t}$ for each $z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}$. Hence,

$$\begin{aligned}&\sup_{g\in\mathcal{G}_d(Q_{t\hat{h}_t})}\int_{\mathbb{R}^d}\eta_t(x)g^2(x)\,dx\\ &\qquad\geq\sup_{g\in\mathcal{G}_d(z+Q_r)}\int_{\mathbb{R}^d}\eta_t(x)g^2(x)\,dx,\qquad z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}.\end{aligned}$$

Thus

$$\sup_{g\in\mathcal{G}_d(Q_{t\hat{h}_t})}\int_{\mathbb{R}^d}\eta_t(x)g^2(x)\,dx\geq\max_{z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}}\sup_{g\in\mathcal{G}_d(z+Q_r)}\int_{\mathbb{R}^d}\eta_t(x)g^2(x)\,dx.\tag{6.6}$$

Let the smooth function $\alpha(\cdot)\colon\mathbb{R}^+\longrightarrow[0,1]$ be given as in Section 3. Given $a>0$, write

$$K_{t,a}(x)=\begin{cases}\dfrac{\alpha(a^{-1}(\hat{h}_t)^{(2+d-p)/(d-p)}|x|)}{|x|^p}, & \text{when applied to (6.1)},\\[2ex] \dfrac{\alpha(a^{-1}(d\log\hat{h}_t)^{1/p}|x|)}{|x|^p}, & \text{when applied to (6.2)},\end{cases}$$

and

$$L_{t,a}(x) = \begin{cases} \dfrac{1-\alpha(a^{-1}(\hat{h}_t)^{(2+d-p)/(d-p)}|x|)}{|x|^p}, & \text{when applied to (6.1)}, \\ \dfrac{1-\alpha(a^{-1}(d\log\hat{h}_t)^{1/p}|x|)}{|x|^p}, & \text{when applied to (6.2)}. \end{cases}$$

By the equality

$$\begin{aligned} &\int_{\mathbb{R}^d} \eta_t(x) g^2(x)\,dx \\ &\quad = (\pm\theta)\hat{h}_t^{p-2}\int_{\mathbb{R}^d}\left[\int_{z+Q_r} K_{a,t}(y-x)g^2(y)\,dy\right][\omega(\hat{h}_t^{-d}\,dx) - \hat{h}_t^{-d}\,dx] \\ &\qquad + (\pm\theta)\hat{h}_t^{p-2}\int_{\mathbb{R}^d}\left[\int_{z+Q_r} L_{a,t}(y-x)g^2(y)\,dy\right][\omega(\hat{h}_t^{-d}dx) - \hat{h}_t^{-d}\,dx] \\ &\quad = \theta\hat{h}_t^{p-2}\big(A_z(g) + B_z(g)\big) \qquad \text{(say)} \end{aligned}$$

and by triangular inequality, the right-hand side of (6.6) is no less than

$$\hat{h}_t^{p-2}\Big\{\max_{z\in h_t^s\mathbb{Z}^d\cap Q_{\hat{h}_t-r}}\ \sup_{g\in\mathcal{G}_d(z+Q_r)} A_z(g) - \max_{z\in h_t^s\mathbb{Z}^d\cap Q_{\hat{h}_t-r}}\ \sup_{g\in\mathcal{G}_d(z+Q_r)} |B_z(g)|\Big\}.$$

In addition, the random variables

$$\sup_{g\in\mathcal{G}_d(z+Q_r)} |B_z(g)|; \qquad z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r},$$

are identically distributed. Therefore, for any $\delta > 0$,

$$\begin{aligned} &\mathbb{P}\Big\{\max_{z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}}\ \sup_{g\in\mathcal{G}_d(z+Q_r)} |B_z(g)| \ge \delta\theta^{-1}\hat{h}_t^{2-p}\Big\} \\ &\qquad \le \#\{h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}\}\mathbb{P}\Big\{\sup_{g\in\mathcal{G}_d(Q_r)} |B_0(g)| \ge \delta\theta^{-1}\hat{h}_t^{2-p}\Big\}. \end{aligned}$$

Further, since $\alpha(\cdot)$ is supported on $[0,3]$ and $s > \frac{2}{d-p}$,

$$A_z(g) = \pm\theta\int_{z+Q_{2^{-1}h_t^s}}\left[\int_{z+Q_r} K_{a,t}(y-x)g^2(y)\,dy\right][\omega(\hat{h}_t^{-d}\,dx) - \hat{h}_t^{-d}\,dx]$$

for all $z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}$ as t is sufficiently large. Consequently, the random variables

$$\sup_{g\in\mathcal{G}_d(z+Q_r)} A_z(g); \qquad z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r},$$

form an i.i.d. sequence. Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\max_{z\in h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}}\sup_{g\in\mathcal{G}_d(z+Q_r)}A_z(g)\le\frac{1+\delta}{\theta}\hat{h}_t^{2-p}\right\}\\
&\quad=\left(\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}A_0(g)\le\frac{1+\delta}{\theta}\hat{h}_t^{2-p}\right\}\right)^{\#\{h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}\}}\\
&\quad=\left(1-\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}A_0(g)\ge\frac{1+\delta}{\theta}\hat{h}_t^{2-p}\right\}\right)^{\#\{h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}\}}.
\end{aligned}$$

Summarizing our argument since (6.5) and (6.6),

$$\begin{aligned}
&\mathbb{P}\{\lambda_{\theta\xi}(Q_t)\le uh_t^2\}\\
&\quad\le\left(1-\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}A_0(g)\ge\frac{1+\delta}{\theta}\hat{h}_t^{2-p}\right\}\right)^{\#\{h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}\}}\\
&\qquad+\#\{h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}\}\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}|B_0(g)|\ge\delta\theta^{-1}\hat{h}_t^{2-p}\right\}.
\end{aligned}\tag{6.7}$$

Once again, we reach the point of using Theorem 3.1 and Theorem 3.2. In connection with (6.1), by definition

$$\begin{aligned}
\sup_{g\in\mathcal{G}_d(Q_r)}A_0(g)&=-\inf_{g\in\mathcal{G}_d(Q_r)}\int_{\mathbb{R}^d}\left[\int_{Q_r}K_{a,t}(y-x)g^2(y)\,dy\right]\\
&\qquad\times[\omega(\hat{h}_t\,dx)-\hat{h}_t^{-d}\,dx],\\
\sup_{g\in\mathcal{G}_d(Q_r)}|B_0(g)|&=\sup_{g\in\mathcal{G}_d(Q_r)}\left|\int_{\mathbb{R}^d}\left[\int_{Q_r}L_{a,t}(y-x)g^2(y)\,dy\right]\right.\\
&\qquad\left.\times[\omega(\hat{h}_t^{-d}\,dx)-\hat{h}_t^{-d}\,dx]\right|.
\end{aligned}$$

Taking $\varepsilon=\hat{h}_t^{-d}$ in (3.7) and (3.6),

$$\begin{aligned}
&\liminf_{a\to\infty}\liminf_{t\to\infty}\frac{1}{\log t}\log\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}A_0(g)\ge\frac{1+\delta}{\theta}\hat{h}_t^{2-p}\right\}\\
&\quad\ge-u^{d/(d-p)}I_{Q_r}\left(\frac{1+\delta}{\theta}\right),\\
&\lim_{a\to\infty}\limsup_{t\to\infty}\frac{1}{\log t}\log\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}|B_0(g)|\ge\delta\theta^{-1}\hat{h}_t^{2-p}\right\}=-\infty,
\end{aligned}$$

where the rate functions $I_{Q_r}(\cdot)$ are defined in (3.9).

By definition,

$$\sup_{g\in\mathcal{G}_d(Q_r)}\|g\|^2_{\mathcal{L}^2(Q_r)}\le 1.$$

We claim that

$$\lim_{r\to\infty}\sup_{g\in\mathcal{G}_d(Q_r)}\|g\|^2_{\mathcal{L}^2(Q_r)}=1. \tag{6.8}$$

Indeed, for a fixed $g\in\mathcal{F}_d(Q_1)$ the function

$$f_r(x)=\left(r^d+\frac{1}{2}r^{d-2}\|\nabla g\|^2_{\mathcal{L}^2(Q_1)}\right)^{-1/2}g\left(\frac{x}{r}\right),\qquad x\in Q_r,$$

is in $\mathcal{G}_d(Q_r)$ and

$$\sup_{g\in\mathcal{G}_d(Q_r)}\|g\|^2_{\mathcal{L}^2(Q_r)}\ge\|f_r\|^2_{\mathcal{L}^2(Q_r)}=\frac{r^d}{r^d+(1/2)r^{d-2}\|\nabla g\|^2_{\mathcal{L}^2(Q_1)}}\longrightarrow 1$$
$$(r\to\infty).$$

By (6.8) and by the definition of $I_{Q_r}(\cdot)$ given in (3.9),

$$\lim_{r\to\infty}I_{Q_r}\left(\frac{1+\delta}{\theta}\right)=\left(\frac{(d-p)(1+\delta)}{d\theta}\right)^{d/(d-p)}\left(\frac{\omega_d}{d}\Gamma\left(\frac{2p-d}{p}\right)\right)^{-p/(d-p)}.$$

Take $u<(1+2\delta)^{-1}\Lambda_0(\theta)$. There is a $\nu(\delta)>0$ such that when a and r are sufficiently large,

$$\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}A_0(g)\ge\frac{1+\delta}{\theta}\hat{h}_t^{2-p}\right\}\ge\exp\{-(d-\nu(\delta))\log t\}=t^{-(d-\nu(\delta))}$$

and

$$\mathbb{P}\left\{\sup_{g\in\mathcal{G}_d(Q_r)}|B_0(g)|\ge\delta\theta^{-1}\hat{h}_t^{2-p}\right\}\le\exp\{-2d\log t\}=t^{-2d}$$

for sufficiently large t.

Being brought to (6.7), our estimates give

$$\begin{aligned}&\mathbb{P}\{\lambda_{\theta\xi}(Q_t)\le uh_t^2\}\\&\quad\le\left(1-t^{-(d-\nu(\delta))}\right)^{\#\{h_t^s\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}\}}+\#\{h_t^s\mathbb{Z}^d\cap Q_{\sqrt{u}th_t-r}\}t^{-2d}\\&\quad\le\exp\{-c_1t^{\nu(\delta)}h_t^{-d(s-1)}\}+c_2t^{-d}.\end{aligned}\tag{6.9}$$

For any $\gamma>1$ and $u<(1+2\delta)^{-1}\Lambda_0(\theta)$, therefore,

$$\sum_k\mathbb{P}\{\lambda_{\theta\xi}(Q_{\gamma^k})\le uh^2_{\gamma^k}\}<\infty.$$

By the Borel–Cantelli lemma,

$$\liminf_{k\to\infty}h_{\gamma^k}^{-2}\lambda_{\theta\xi}(Q_{\gamma^k})\ge(1+2\delta)^{-1}\Lambda_0(\theta)\qquad\text{a.s.}$$

Since $\lambda_{\theta\xi}(Q_t)$ is monotonic in t and $\delta>0$ can be arbitrarily small, we have proved (6.4) associated with (6.1).

As for (6.2), by (3.11) and (3.12) (with $\varepsilon = \hat{h}_t^{-d}$),

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\Big\{ \sup_{g\in\mathcal{G}_d(Q_r)} |B_0(g)| \geq \delta\theta^{-1}\hat{h}_t^{2-p} \Big\} = -\infty,$$

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\Big\{ \sup_{g\in\mathcal{G}_d(Q_r)} A_0(g) \geq \frac{1+\delta}{\theta}\hat{h}_t^{2-p} \Big\}$$
$$= -u^{d/(d-p)} \frac{2+d-p}{(2-p)\rho^*_{Q_r}} \frac{1+\delta}{\theta} \geq -u^{d/(d-p)} \frac{2+d-p}{(2-p)\rho_{Q_r}} \frac{1+\delta}{\theta},$$

where ρ^*_D is defined in (3.10) and

$$\rho_{Q_r} = \sup_{g\in\mathcal{G}_d(Q_r)} \int_{Q_r} \frac{g^2(x)}{|x|^p}\, dx.$$

Clearly, ρ_{Q_r} is nondecreasing in r and $\rho_{Q_r} \leq \rho(d,p)$, where $\rho(d,p)$ is defined in (5.14). We claim that

$$\lim_{r\to\infty} \rho_{Q_r} = \rho(d,p). \tag{6.10}$$

Indeed, let $\alpha(\cdot)$ be the smooth truncation function introduced in Section 3. For any $f \in \mathcal{G}_d(\mathbb{R}^d)$, write

$$f_r(x) = f(x)\alpha(3r^{-1}|x|).$$

The function

$$g_r(x) = \big(\|f_r\|_{\mathcal{L}^2(Q_r)} + 2^{-1}\|\nabla f_r\|_{\mathcal{L}^2(Q_r)}\big)^{-1/2} f_r(x) \tag{6.11}$$

is in $\mathcal{G}_d(Q_r)$. Thus, by the fact that $\alpha(\cdot) \geq 1_{[0,1]}(\cdot)$

$$\rho_{Q_r} \geq \int_{Q_r} \frac{g_r^2(x)}{|x|^p}\, dx \geq \bigg(\|f_r\|^2_{\mathcal{L}^2(Q_r)} + \frac{1}{2}\|\nabla f_r\|^2_{\mathcal{L}^2(Q_r)}\bigg)^{-1} \int_{\{|x|\leq r/3\}} \frac{f^2(x)}{|x|^p}\, dx.$$

Notice that $\|f_r\|^2_{\mathcal{L}^2(Q_r)} \leq \|f\|_2^2$ and

$$|\nabla f_r(x)| \leq 3r^{-1}|\alpha'(3r^{-1}|x|)| \cdot |f(x)| + \alpha(3r^{-1}|x|)|\nabla f(x)|$$
$$\leq 3r^{-1}|f(x)| + |\nabla f(x)|,$$

where the last step follows from the fact that $|\alpha(\cdot)| \leq 1$ and $|\alpha'(\cdot)| \leq 1$.

Thus,

$$\liminf_{r\to\infty} \big(\|f_r\|^2_{\mathcal{L}^2(Q_r)} + \tfrac{1}{2}\|\nabla f_r\|^2_{\mathcal{L}^2(Q_r)}\big)^{-1} \geq \big(\|f\|_2^2 + \tfrac{1}{2}\|\nabla f\|_2^2\big)^{-1} = 1. \tag{6.12}$$

Summarizing our argument,

$$\liminf_{r\to\infty} \rho_{Q_r} \geq \int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^p}\, dx.$$

Taking supremum over $f \in \mathcal{G}_d$ on the right-hand side leads to (6.10).

By (6.10) and (A.7), therefore,

$$\lim_{r\to\infty} \rho_{Q_r} = \left(\frac{2-p}{2}\right)^{(2-p)/2} p^{p/2}\sigma(d, p).$$

Similarly, the above discussion leads to (6.4) [corresponding to (6.2)], again by the Borel–Cantelli lemma.

APPENDIX

LEMMA A.1. *Under $d/2 < p < d$,*

$$\int_{\mathbb{R}^d} \left[\exp\left\{-\frac{1}{|x|^p}\right\} - 1 + \frac{1}{|x|^p}\right] dx = \omega_d \frac{p}{d-p} \Gamma\left(\frac{2p-d}{p}\right), \tag{A.1}$$

where ω_d is the volume of the d-dimensional unit ball.

PROOF. By the sphere substitution,

$$\begin{aligned}\int_{\mathbb{R}^d} \left[\exp\left\{-\frac{1}{|x|^p}\right\} - 1 + \frac{1}{|x|^p}\right] dx &= d\omega_d \int_0^\infty \left[\exp\left\{-\frac{1}{\rho^p}\right\} - 1 + \frac{1}{\rho^p}\right] \rho^{d-1}\, d\rho \\ &= \frac{d\omega_d}{p} \int_0^\infty [e^{-\gamma} - 1 + \gamma]\gamma^{-(d+p)/p}\, d\gamma,\end{aligned}$$

where the second step follows from the substitution $\rho = \gamma^{-1/p}$.

Applying the integration by parts twice (under the assumption $d/2 < p < d$),

$$\begin{aligned}\int_0^\infty [e^{-\gamma} - 1 + \gamma]\gamma^{-(d+p)/p}\, d\gamma &= \frac{p}{d} \int_0^\infty [1 - e^{-\gamma}]\gamma^{-d/p}\, d\gamma \\ &= \frac{p^2}{d(d-p)} \int_0^\infty \gamma^{-(d-p)/d} e^{-\gamma}\, d\gamma \\ &= \frac{p^2}{d(d-p)} \Gamma\left(\frac{2p-d}{p}\right).\end{aligned}$$

We have proved identity (A.1). □

Recall that for any domain $D \subset \mathbb{R}^d$,

$$\mathcal{G}_d(D) = \{g \in W^{1,2}(D); \|g\|^2_{\mathcal{L}^2(D)} + \tfrac{1}{2}\|\nabla g\|^{-2}_{\mathcal{L}^2(D)} = 1\},$$

$$\mathcal{F}_d(D) = \{g \in W^{1,2}(D); \|g\|_{\mathcal{L}^2(D)} = 1\}.$$

In particular, $\mathcal{G}_d = \mathcal{G}_d(\mathbb{R}^d)$ and $\mathcal{F}_d = \mathcal{F}_d(\mathbb{R}^d)$.

LEMMA A.2. *Let the functional* $Z(g^2)$ $[g \in W^{1,2}(D)]$ *satisfy* $Z(cg^2) = cZ(g^2)$ *for every* $g \in W^{1,2}(D)$ *and* $c > 0$. *Then*

$$\sup_{g\in\mathcal{F}_d(D)}\left\{Z(g^2) - \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right\} > 1$$

if any only if $\sup_{g\in\mathcal{G}_d(D)} Z(g^2) > 1$.

PROOF. For any $g \in \mathcal{F}_d(D)$,

$$Z(g^2) \le \Big(\sup_{f\in\mathcal{G}_d(D)} Z(f^2)\Big)\left(1 + \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right).$$

Hence,

$$\begin{aligned}&\sup_{g\in\mathcal{F}_d(D)}\left\{Z(g^2) - \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right\}\\ &\quad\le \sup_{g\in\mathcal{F}_d(D)}\left\{\Big(\sup_{f\in\mathcal{G}_d(D)} Z(f^2)\Big)\left(1 + \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right) - \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right\}.\end{aligned}$$

Therefore, $\sup_{g\in\mathcal{G}_d(D)} Z(g^2) > 1$, if

$$\sup_{g\in\mathcal{F}_d(D)}\left\{Z(g^2) - \frac{1}{2}\int_D |\nabla g(x)|^2\,dx\right\} > 1.$$

On the other hand, assume $\sup_{g\in\mathcal{G}_d(D)} Z(g^2) > 1$. Then there is $g_0 \in \mathcal{G}_d(D)$ such that $Z(g_0^2) > 1$. Write $f_0(x) = g_0(x)/\|g_0\|_{\mathcal{L}^2(D)}$. We have $f_0 \in \mathcal{F}_d(D)$ and

$$Z(f_0^2) - \frac{1}{2}\int_D |\nabla f_0(x)|^2\,dx > \|g_0\|^{-2}_{\mathcal{L}^2(D)} - \|g_0\|^{-2}_{\mathcal{L}^2(D)}\big(1 - \|g_0\|^2_{\mathcal{L}^2(D)}\big) = 1. \qquad \square$$

It was shown (see [1], (1.19)) that for every $\lambda > 0$,

$$M(\lambda) \equiv \sup_{g\in\mathcal{F}_d}\left\{\lambda\int_{\mathbb{R}^d} \frac{g^2(x)}{|x|^p}\,dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2\,dx\right\} < \infty \tag{A.2}$$

under $d/2 < p < \min\{2, d\}$.

Further, by rescaling $g(x) \mapsto a^{d/2}g(ax)$ for suitable $a > 0$, one can show that

$$M(\lambda) = \lambda^{2/(2-p)}M(1). \tag{A.3}$$

LEMMA A.3. *Under* $d/2 < p < \min\{2, d\}$, *there is a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^p}\,dx \le C\|f\|_2^{2-p}\|\nabla f\|_2^p \qquad \forall f \in W^{1,2}(\mathbb{R}^d). \tag{A.4}$$

Further, let $\sigma(d,p)$ be the smallest (infimum) among above C. Then

$$M(\lambda)=\frac{2-p}{2}p^{p/(2-p)}(\lambda\sigma(d,p))^{2/(2-p)},\qquad \lambda>0. \tag{A.5}$$

In addition,

$$\rho(d,p)\equiv\sup\left\{\int_{\mathbb{R}^d}\frac{g^2(x)}{|x|^p}\,dx;\,g\in\mathcal{G}_d\right\}<\infty \tag{A.6}$$

and

$$\rho(d,p)=\left(\frac{2-p}{2}\right)^{(2-p)/2}p^{p/2}\sigma(d,p). \tag{A.7}$$

PROOF. In view of (A.3) we may take $\lambda=1$ in (A.5). For any $f\in W^{1,2}$ with $\|f\|_2=1$, let

$$\int_{\mathbb{R}^d}\frac{f^2(x)}{|x|^p}\,dx=C_f\|\nabla f\|_2^p.$$

Given $\gamma>0$, let $g(x)=\gamma^{d/2}f(\gamma x)$. Then $\|g\|_2=1$, $\|\nabla g\|_2=\gamma\|\nabla f\|_2$, and therefore

$$\int_{\mathbb{R}^d}\frac{g^2(x)}{|x|^p}\,dx=\gamma^p\int_{\mathbb{R}^d}\frac{f^2(x)}{|x|^p}\,dx=\gamma^pC_f\|\nabla f\|_2^p=C_f\|\nabla g\|_2^p.$$

Thus

$$M(1)\ge C_f\|\nabla g\|_2^p-\tfrac{1}{2}\|\nabla g\|_2^2=C_f\gamma^p\|\nabla f\|_2^p-\tfrac{1}{2}\gamma^2\|\nabla f\|_2^2.$$

Since $\gamma>0$ is arbitrary, the variable $\gamma\|\nabla f\|_2$ runs over all positive numbers. Consequently,

$$M(1)\ge\sup_{x>0}\left\{C_fx^p-\frac{1}{2}x^2\right\}=\frac{2-p}{2}C_f^{2/(2-p)}p^{p/(2-p)}.$$

By homogeneity, we have proved (A.4) with

$$M(1)\ge\frac{2-p}{2}p^{p/(2-p)}\sigma(d,p)^{2/(2-p)}.$$

On the other hand, for any $g\in\mathcal{F}_d$

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{g^2(x)}{|x|^p}\,dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx&\le\sigma(d,p)\|\nabla g\|_2^p-\frac{1}{2}\|\nabla g\|_2^2\\&\le\sup_{x>0}\left\{\sigma_1(d,p)x^p-\frac{1}{2}x^2\right\}\\&=\frac{2-p}{2}p^{p/(2-p)}\sigma(d,p)^{2/(2-p)}.\end{aligned}$$

We have proved (A.5).

Obviously, (A.6) follows from (A.4). Take

$$Z(g^2) = \frac{1}{\rho(d,p)} \int_{\mathbb{R}^d} \frac{g^2(x)}{|x|^p}\,dx, \qquad g \in W^{1,2}(\mathbb{R}^d).$$

We have that $\sup_{g\in\mathcal{G}_d} Z(g^2) = 1$. By (A.3), the function $M(\lambda)$ is continuous and increasing. By Lemma A.2, we must have

$$M\left(\frac{1}{\rho(d,p)}\right) = 1. \tag{A.8}$$

Finally, (A.7) follows from (A.5) and (A.8). □

Another variation appearing in this paper is

$$\rho^*(d,p) = \sup_{g\in\mathcal{G}_d} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g^2(y)}{|y-x|^p}\,dy.$$

We now claim that

$$\rho^*(d,p) = \rho(d,p). \tag{A.9}$$

Indeed,

$$\begin{aligned}\rho^*(d,p) &= \sup_{x\in\mathbb{R}^d} \sup_{g\in\mathcal{G}_d} \int_{\mathbb{R}^d} \frac{g^2(y)}{|y-x|^p}\,dy = \sup_{x\in\mathbb{R}^d} \sup_{g\in\mathcal{G}_d} \int_{\mathbb{R}^d} \frac{g_x^2(y)}{|y|^p}\,dy \\ &\le \sup_{g\in\mathcal{G}_d} \int_{\mathbb{R}^d} \frac{g^2(y)}{|y|^p}\,dy = \rho(d,p),\end{aligned}$$

where $g_x(y) = g(x+y)$, and the inequality follows from the fact that $g_x \in \mathcal{G}_d$ as soon as $g \in \mathcal{G}_d$.

Acknowledgment. The author would like to thank the anonymous referee who read the first version of this paper for his/her interesting remarks and excellent suggestions.

REFERENCES

[1] BASS, R., CHEN, X. and ROSEN, J. (2009). Large deviations for Riesz potentials of additive processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 626–666. MR2548497

[2] BISKUP, M. and KÖNIG, W. (2001). Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* **29** 636–682. MR1849173

[3] CADEL, A., TINDEL, S. and VIENS, F. (2008). Sharp asymptotics for the partition function of some continuous-time directed polymers. *Potential Anal.* **29** 139–166. MR2430611

[4] CARMONA, R., VIENS, F. G. and MOLCHANOV, S. A. (1996). Sharp upper bound on the almost-sure exponential behavior of a stochastic parabolic partial differential equation. *Random Oper. Stoch. Equ.* **4** 43–49. MR1393184

[5] CARMONA, R. A. and MOLCHANOV, S. A. (1994). *Parabolic Anderson Problem and Intermittency*. *Mem. Amer. Math. Soc.* **108**. Amer. Math. Soc., Providence, RI. MR1185878

[6] CARMONA, R. A. and MOLCHANOV, S. A. (1995). Stationary parabolic Anderson model and intermittency. *Probab. Theory Related Fields* **102** 433–453. MR1346261

[7] CARMONA, R. A. and VIENS, F. G. (1998). Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter. *Stoch. Stoch. Rep.* **62** 251–273. MR1615092

[8] CHEN, X. (2010). *Random Walk Intersections*: *Large Deviations and Related Topics*. *Math. Surveys Monogr.* **157**. Amer. Math. Soc., Providence, RI. MR2584458

[9] CHEN, X. and KULIK, A. M. (2010). Brownian motion and parabolic Anderson model in a renormalized Poisson potential. *Ann. Inst. Henri Poincaré*. To appear.

[10] CHEN, X. and ROSINSKI, J. (2011). Spatial Brownian motion in renormalized Poisson potential: A critical case. Preprint.

[11] CRANSTON, M., MOUNTFORD, T. S. and SHIGA, T. (2005). Lyapunov exponent for the parabolic Anderson model with Lévy noise. *Probab. Theory Related Fields* **132** 321–355. MR2197105

[12] DALANG, R. C. and MUELLER, C. (2009). Intermittency properties in a hyperbolic Anderson problem. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 1150–1164. MR2572169

[13] ENGLÄNDER, J. (2008). Quenched law of large numbers for branching Brownian motion in a random medium. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 490–518. MR2451055

[14] FLORESCU, I. and VIENS, F. (2006). Sharp estimation of the almost-sure Lyapunov exponent for the Anderson model in continuous space. *Probab. Theory Related Fields* **135** 603–644. MR2240702

[15] FUKUSHIMA, R. (2010). Second order asymptotics for Brownian motion among a heavy tailed Poissonian potential. Preprint.

[16] GÄRTNER, J. and KÖNIG, W. (2000). Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10** 192–217. MR1765208

[17] GÄRTNER, J., KÖNIG, W. and MOLCHANOV, S. A. (2000). Almost sure asymptotics for the continuous parabolic Anderson model. *Probab. Theory Related Fields* **118** 547–573. MR1808375

[18] GÄRTNER, J. and MOLCHANOV, S. A. (1990). Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.* **132** 613–655. MR1069840

[19] GÄRTNER, J. and MOLCHANOV, S. A. (1998). Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. *Probab. Theory Related Fields* **111** 17–55. MR1626766

[20] HARVLIN, S. and AVRAHAM, B. (1987). Diffusion in disordered media. *Adv. in Phys.* **36** 695–798.

[21] KOMOROWSKI, T. (2000). Brownian motion in a Poisson obstacle field. *Astérisque* **266** 91–111. MR1772671

[22] SZNITMAN, A.-S. (1993). Brownian asymptotics in a Poissonian environment. *Probab. Theory Related Fields* **95** 155–174. MR1214085

[23] SZNITMAN, A.-S. (1993). Brownian survival among Gibbsian traps. *Ann. Probab.* **21** 490–508. MR1207235

[24] SZNITMAN, A.-S. (1998). *Brownian Motion*, *Obstacles and Random Media*. Springer, Berlin. MR1717054

[25] VAN DER HOFSTAD, R., KÖNIG, W. and MÖRTERS, P. (2006). The universality classes in the parabolic Anderson model. *Comm. Math. Phys.* **267** 307–353. MR2249772

[26] VAN DER HOFSTAD, R., MÖRTERS, P. and SIDOROVA, N. (2008). Weak and almost sure limits for the parabolic Anderson model with heavy tailed potentials. *Ann. Appl. Probab.* **18** 2450–2494. MR2474543

[27] VIENS, F. G. and ZHANG, T. (2008). Almost sure exponential behavior of a directed polymer in a fractional Brownian environment. *J. Funct. Anal.* **255** 2810–2860. MR2464192

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE 37996
USA
E-MAIL: xchen@math.utk.edu