Midterm Exam **Math 523 (Fall 2022)** **Name:**

1. Let A, B and C be three events. Prove:

(1). $1_{A\Delta B} = |1_A - 1_B|$

Proof. Both side equal 0 or 1. $1_{A\Delta B} = 1$ if and only if $\omega \in A \setminus B$ or $\omega \in B \setminus A$. In either case, there is exactly one of 1_A and 1_B is one and another is 0. In either case $|1_A - 1_B| = 1$.

(2). $P(A\Delta C) \le P(A\Delta B) + P(B\Delta C)$ (triangle inequality).

Proof. By triangle inequality

$$|1_A - 1_C| = |(1_A - 1_B) + (1_B - 1_C)| \le |1_A - 1_B| + |1_B - 1_C|$$

By part (1), it is re-written as $1_{A\Delta C} \le 1_{A\Delta B} + 1_{B\Delta C}$. Hence,

$$P(A\Delta C) \le E1_{A\Delta C} \le E(1_{A\Delta B} + 1_{B\Delta C}) = E1_{A\Delta B} + E1_{B\Delta C} = P(A\Delta B) + P(B\Delta C)$$

2. In view of Problem 14 in Section 1.6 (p.24), the father-kid matching appears to be a complicated problem. Let X_n be the matching number among n fathers and n childrens. Find EX_n.

Solution. Label the fathers by $1, 2, \cdots, n$ according to the time they arrive. Set

$$Y_k = \begin{cases} 1 & \text{if the father "}k\text{" picks his child} \\ 0 & \text{otherwise} \end{cases}$$

Then $X_n = Y_1 + \cdots + Y_n$. Set A_k as the event that his child has not been picked by other earlier arrived fathers.

$$P\{Y_k = 1\} = P(A_k)P\{Y_k = 1|A_k\} = \frac{(n-1)\cdots(n-k)}{n\cdots(n-k+1)}\frac{1}{(n-k)} = \frac{n-k}{n}\cdot\frac{1}{n-k} = \frac{1}{n}$$

Hence,

$$EY_k = P\{X_k = 1\} = \frac{1}{n} \qquad k = 1, \cdots, n$$

Thus,

$$EX_n = E(Y_1 + \cdots + Y_n) = EY_1 + \cdots + EY_n = 1$$

3. Let $\{A_n\}_{n\ge 1}$ be a sequence of events such that $P(A_n) \longrightarrow 0$ as $n \to \infty$. Prove that there is a subsequence $\{n_k\}$ of positive integers such that

$$P\Big\{A_{n_k} \text{ occurs infinitely often}\Big\} = 0$$

Proof. Pick n_k such that

$$\sum_{k=1}^{\infty} P(A_{n_k}) < \infty$$

Notice

$$\Big\{A_{n_k} \quad \text{occurs infinitely often}\Big\} = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_{n_k}$$

By continuity theorem

$$P\Big\{A_{n_k} \quad \text{occurs infinitely often}\Big\} = P\Big(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_{n_k}\Big) = \lim_{m\to\infty} P\Big(\bigcup_{k=m}^{\infty} A_{n_k}\Big)$$

By sub-additivity

$$P\Big(\bigcup_{k=m}^{\infty} A_{n_k}\Big) \le \sum_{k=m}^{\infty} P(A_{n_k})$$

The right hand side converges to 0 as $m \to \infty$. So we have

$$P\Big\{A_{n_k} \quad \text{occurs infinitely often}\Big\} = 0$$

4. Sometimes we consider extended random vairable (Line 2, p.26). That is, a random variable is a measurable map $X\colon\ \Omega \longrightarrow [-\infty, \infty]$. Under this definition, X is allowed to take $\pm\infty$. Staying with this new definiton, prove that $|X| < \infty$ a.s. if and only if

$$\lim_{n\to\infty} P\{|X| \ge n\} = 0$$

Proof. Notice that

$$\{|X| = \infty\} = \bigcap_{n=1}^{\infty} \{|X| \ge n\}$$

By continuity theorem,

$$P\{|X| = \infty\} = P\Big(\bigcap_{n=1}^{\infty} \{|X| \ge n\}\Big) = \lim_{n\to\infty} P\{|X| \ge n\}$$

Hence, $|X| < \infty$ a.s. if and only if the right hand side is equal to 0.

5. Let X be a non-negative random variable on the probability space $(\Omega, \mathcal{F}, P)$ with $EX < \infty$. Let $A \in \mathcal{F}$ be an event such that

$$P\big(A \cap \{X \le x\}\big) = P(A)P\{X \le x\} \qquad \forall x \in \mathbf{R}$$

Prove that

$$EX1_A = P(A)EX$$

Proof. First, for any $a < b$,

$$
\begin{aligned}
&P\big(A\cap\{a < X \le b\}\big) = P\big(A\cap\{X \le b\}\big) - P\big(A\cap\{X \le a\}\big)\\
&= P(A)P\{X \le b\} - P(A)P\{P\{X \le a\} = P(A)\Big(P\{X \le b\} - P\{X \le a\}\Big)\\
&= P(A)P\{a < X \le b\}
\end{aligned}
$$

Define the non-negative simple random variables

$$X_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1\{\frac{k-1}{2^n} < X \le \frac{k}{2^n}\} \qquad n-1, 2, \cdots$$

Then,

$$
\begin{aligned}
&EX_n1_A = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} E1\{\frac{k-1}{2^n} < X \le \frac{k}{2^n}\}1_A = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\Big(A\cap\{\frac{k-1}{2^n} < X \le \frac{k}{2^n}\}\Big)\\
&= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P(A)P\{\frac{k-1}{2^n} < X \le \frac{k}{2^n}\} = P(A)\sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\{\frac{k-1}{2^n} < X \le \frac{k}{2^n}\}\\
&= P(A)EX_n
\end{aligned}
$$

We now let $n\to\infty$ on the both sides. Notice that$X_n \uparrow X$ as $n\to\infty$. So by the definition of EX for non-negative X,

$$\lim_{n\to\infty} EX_n = EX$$

Further, by the fact $Z_n = X_n1_A$ is non-negative simple simple with $Z_n \uparrow X1_A$ $(n\to\infty)$, we have

$$\lim_{n\to\infty} EX_n1_A = EX1_A$$

Another alternative justification for taking the limit $n\to\infty$ is to apply mononotonic convergence to X_n and X_n1_A separately.