Final Exam **Math 523 (Fall 2022)**

1. (10 points) Let the random variable X satisfy $EX^2 < \infty$. Compute

$$\lim_{n\to\infty} nEX \sin\frac{X}{n}$$

and justify your steps.

Solution First notice

$$nEX\sin\frac{X}{n} = EnX\sin\frac{X}{n}$$

and

$$\lim_{n\to\infty} nX\sin\frac{X}{n} = X^2$$

By the fact that

$$|nX\sin\frac{X}{n}| = n|X|\cdot|\sin\frac{X}{n}| \le n|X|\Big|\frac{X}{n}\Big| = X^2 \qquad n = 1, 2, \cdots$$

and by Dominated convergence

$$\lim_{n\to\infty} nEX\sin\frac{X}{n} = E\lim_{n\to\infty} nX\sin\frac{X}{n} = EX^2$$

2. (15 points). Recall (Problem #8, p.115) that under the assumption

$$\sum_{n=1}^{\infty} P\{X_n > A\} < \infty \qquad \text{for some } A > 0$$

the independent sequence $\{X_n\}_{n\ge 1}$ satisfies

$$X \triangleq \sup_{n\ge 1} X_n < \infty \qquad a.s.$$

Consequently,

$$Y \triangleq \limsup_{n\to\infty} X_n < \infty \qquad a.s.$$

Is X a constant almost surely? How about Y? Please prove your conclusion if it is "Yes", and provide a counter example if your answer is "No".

Solution. X does not have to be constant: Take X_1 to be a non-negative and non-degenrate random variable ($X \sim \exp(1)$, for example) and take $X_n = 0$ for $n \ge 2$. Then $X_1, X_2, \cdots$ are independent sequence and $X = X_1$

Y is a constant almost surely. Indeed, Y is measurable with respect to $\mathcal{F}_m \triangleq \sigma(X_m, X_{m+1}, \cdots)$ for any $m \geq 1$ and therefore $\mathcal{F}_\infty$-measurable with

$$\mathcal{F}_\infty = \bigcap_{m=1}^{\infty} \mathcal{F}_m$$

The claim follows from Kolmogorov 0-1 law (Corollary 10.1, p.73).

3. (15 points). Prove that for any random variable X,

$$E\exp\{X\} < \infty$$

if and only if

$$\sum_{n=1}^{\infty} P\{X \geq \log n\} < \infty$$

Proof. Notice $\exp\{X\} \geq 0$. By Theorem 12.1-(iii), p.75

$$E\exp\{X\} < \infty$$

if and only if

$$\sum_{n=1}^{\infty} P\Big\{\exp\{X\} \geq n\Big\} < \infty$$

The conclusion follows from tha e fact that

$$P\Big\{\exp\{X\} \geq n\Big\} = P\{X \geq \log n\} \qquad \forall n \geq 3$$

4. (10 points). Given two independent random variables X and Y with $EX = 1$, prove that

$$E\exp\Big\{XY\Big\} \geq E\exp\{Y\}$$

Proof. By independence and by Fubini's theorem

$$E\exp\Big\{XY\Big\} = E\Big\{E^X \exp\Big\{XY\Big\}\Big\}$$

By Jensen's inequality

$$E^X \exp\Big\{XY\Big\} \geq \exp\Big\{E^X(XY)\Big\} = \exp\Big\{YE(X)\Big\} = \exp\{Y\}$$

So we have

$$E\exp\Big\{XY\Big\} \geq E\exp\{Y\}$$

5. (20 points). For an i.i.d. sequence $\{X, X_n\}_{n\geq 1}$ and an integer $n \geq 1$, the random variable

$$\bar{X}_n \triangleq \frac{X_1 + \cdots + X_n}{n}$$

is known as the sample mean (of size n) of X. Assume that

$$E\exp\{\theta X\} < \infty \qquad \forall \theta > 0$$

Prove that for any $x, \theta > 0$,

$$P\{\bar{X}_n \geq x\} \leq \Big(e^{-\theta x} E\exp\{\theta X\}\Big)^n$$

Use this bound to show that when $X \sim N(0,1)$,

$$P\{\bar{X}_n \geq x\} \leq \exp\Big\{-\frac{nx^2}{2}\Big\} \qquad \forall x > 0$$

Proof. Notice

$$P\{\bar{X}_n \geq x\} = P\{\theta(X_1 + \cdots + X_n) \geq n\theta x\} = P\Big\{\exp\big\{\theta(X_1 + \cdots + X_n)\big\} \geq \exp\{n\theta x\}\Big\}$$

By Markov inequality

$$\begin{aligned} &P\Big\{\exp\big\{\theta(X_1 + \cdots + X_n)\big\} \geq \exp\{n\theta x\}\Big\} \\ &\leq \exp\{-n\theta x\} E\exp\big\{\theta(X_1 + \cdots + X_n)\big\} \\ &= \exp\{-n\theta x\}\Big(E\exp\big\{\theta X\big\}\Big)^n = \Big(e^{-\theta x} E\exp\{\theta X\}\Big)^n \end{aligned}$$

where the second step follows from independence.

In the setting of $X \sim N(0,1)$,

$$E\exp\{\theta X\} = \exp\Big\{\frac{\theta^2}{2}\Big\}$$

So we have

$$P\{\bar{X}_n \geq x\} \leq \exp\Big\{-n\Big(\theta x - \frac{\theta^2}{2}\Big)\Big\}$$

for any $\theta > 0$. Picking $\theta = x$ we have

$$P\{\bar{X}_n \geq x\} \leq \exp\Big\{-\frac{nx^2}{2}\Big\} \qquad \forall x > 0$$

6. (15 points). Let $\{X_k\}_{k\geq 1}$ be an i.i.d. sequence with common distribution $U(0,1)$ (uniform distribution on $[0,1]$). Prove that the random sequence

$$Z_n \triangleq n \min_{1\leq k\leq n} X_k \qquad n = 1, 2, \cdots$$

converges in distribution as $n \to \infty$, and identify the limit distribution.

Proof. Notice that $Z_n \geq 0$. So we need only consider $x > 0$ in the following computation:

$$F_{Z_n}(x) = P\Big\{n \min_{1\leq k\leq n} X_k \leq x\Big\} = 1 - P\Big\{\min_{1\leq k\leq n} X_k > \frac{x}{n}\Big\}$$
$$= 1 - \Big(P\Big\{X > \frac{x}{n}\Big\}\Big)^n = 1 - \Big(1 - \frac{x}{n}\Big)^n$$

By the fact that

$$\lim_{n\to\infty}\Big(1 - \frac{x}{n}\Big)^n = e^{-x} \qquad \forall x > 0$$

Consequently,

$$\lim_{n\to\infty} F_{Z_n}(x) = 1 - e^{-x} \qquad \forall x > 0$$

Thus, $Z_n \xrightarrow{d} \exp(1)$

7. (15 points) Can the convergence in probability stated in Problem #1 in p.259 be up-graded to almost sure convergence? To receive credit, you have to prove your claim.

Solution. The answer is "Yes". Indeed, for each $\epsilon > 0$

$$P\{Z_n \geq a + \epsilon\} = \Big(P\{X \geq a + \epsilon\}\Big)^n = \Big(\int_{a+\epsilon}^{\infty} e^{-(x-a)}dx\Big)^n = e^{-n\epsilon}$$

Hence

$$\sum_{n=1}^{\infty} P\{|Z_n - a| \geq \epsilon\} = \sum_{n=1}^{\infty} P\{Z_n \geq a + \epsilon\} = \sum_{n=1}^{\infty} e^{-n\epsilon} < \infty$$

By Borel-Cantelli lemma,

$$\limsup_{n\to\infty} |Z_n - a| \leq \epsilon \qquad a.s.$$

Letting $\epsilon \to 0^+$ on the right hand side we have

$$\lim_{n\to\infty} Z_n = a \qquad a.s.$$

Alternative solution By your homework, $Z_n - a \xrightarrow{P} 0$ Notice that $Z_n - a$ is non-negative and non-increasing. So we have $Z_n - a \xrightarrow{a.s.} 0$.