Homework # 4

Section 3.10

2.(a) Write $X^2 = X^{4/3} \cdot X^{2/3}$. Take $p = 3$ and $q = 3/2$ in Hölder inequality

$$1 = EX^2 \le \Big\{EX^4\Big\}^{1/3}\Big\{E|X|\Big\}^{2/3}$$

This leads to the desired inequality.

(b). Assume $EX^2 = 1$. To see how to link the Hölder inequality to our inequality Write $X^2 = X^\alpha X^\beta$ and by Hölder inequality

$$1 = EX^2 = EX^\alpha X^\beta \le \Big\{E|X|^{p\alpha}\Big\}^{1/p}\Big\{E|X|^{q\beta}\Big\}^{1/q}$$

We make $p\alpha = 2m$, $q\beta = 1$, $\alpha + \beta = 2$ and $p^{-1} + q^{-1} = 1$. Solve this system we have

$$\alpha = \frac{2m}{2m-1}, \quad \beta = \frac{2m-2}{2m-1}, \quad p = 2m-1 \quad \text{and} \quad q = \frac{2m-1}{2m-2}$$

So we have

$$1 \le \Big\{EX^{2m}\Big\}^{\frac{1}{2m-1}}\Big\{E|X|\Big\}^{\frac{2m-2}{2m-1}}$$

Or,

$$E|X| \ge \frac{1}{\sqrt[2m-2]{EX^{2m}}}$$

5. By Cauchy-Schwartz inequality

$$EX = EX1_{\{X<\lambda EX\}} + EX1_{\{X\ge\lambda EX\}} \le \lambda EX + \Big\{EX^2\Big\}^{1/2}\Big\{P\{X \ge \lambda EX\}\Big\}^{1/2}$$

This leads to

$$P\{X \ge \lambda EX\} \ge (1-\lambda)^2 \frac{\big(EX\big)^2}{EX^2}$$

7(a). Let $\theta \ge 0$ be a constant.

$$P\big\{X - EX \ge x\big\} = P\big\{(X - EX) + \theta \ge x + \theta\big\}$$
$$\le P\Big\{\big((X-EX)+\theta\big)^2 \ge (x+\theta)^2\Big\} \le \frac{1}{(x+\theta)^2} E\big((X-EX)+\theta\big)^2$$

where the last step follows from Markov inequality. Notice that

$$E\big((X-EX)+\theta\big)^2 = E(X-EX)^2 + 2\theta E(X-EX) + \theta^2 = \sigma^2 + \theta^2$$

So we have

$$P\{X - EX \ge x\} \le \frac{\sigma^2 + \theta^2}{(x+\theta)^2}$$

Taking $\theta = \sigma^2/x$ to minimize the right hand side

$$P\{X - EX \ge x\} \le \frac{\sigma^2}{x^2 + \sigma^2}$$

Further,

$$P\{|X - EX| \ge x\} = P\{X - EX \ge x\} + P\{-(X - EX) \ge x\}$$

The first term is bounded by $\frac{\sigma^2}{x^2+\sigma^2}$. Replace X by $-X$ in the first inequality,

$$P\{-(X - EX) \ge x\} \le \frac{\sigma^2}{x^2 + \sigma^2}$$

In summary,

$$P\{|X - EX| \ge x\} \le \frac{2\sigma^2}{x^2 + \sigma^2}$$

Section 5.14

5. By the assumption $\sup\{x; F(x) < 1\} = \infty$ we have $p_t > 0$ for all $t > 0$. Set $Y_t = p_t\tau(t)$. Need to show

$$\lim_{t\to\infty} F_{Y_t}(x) = 1 - e^{-x} \qquad x > 0$$

Let $x > 0$ be fixed. Notice that $\tau(t)$ takes positive integer values

$$F_{Y_t}(x) = P\{\tau(t) \le p_t^{-1}x\} = P\{\tau(t) \le [p_t^{-1}x]\} = P\Big\{\max_{1\le k\le[p_t^{-1}x]} X_k > t\Big\}$$

$$= 1 - P\Big\{\max_{1\le k\le[p_t^{-1}x]} X_k \le t\Big\} = 1 - \Big(P\{X \le t\}\Big)^{[p_t^{-1}x]} = 1 - (1-p_t)^{[p_t^{-1}x]}$$

Notice that $p_t \to 0^+$ as $t \to \infty$. So our assertion follows from the fact that

$$\lim_{t\to\infty} (1-p_t)^{[p_t^{-1}x]} = e^{-x}$$

11. By 0-1 law $P\{X_n \text{ converges}\} = 0$ or 1. We use argument by contradiction: Assume X_n a.s. converges. By 0-1 law again, the limit must be a deterministic constant

$C \in \mathbf{R}$, i.e., $X_n \xrightarrow{a.s.} C$. By the second Borel-Cantelli lemma (Theorem 18.2, p.97) and independence,

$$\sum_{n=1}^{\infty} P\{|X_n - C| \geq \epsilon\} < \infty \qquad \forall \epsilon > 0$$

Or,

$$\sum_{n=1}^{\infty} P\{|X_1 - C| \geq \epsilon\} < \infty \qquad \forall \epsilon > 0$$

So we must have that $P\{|X_1 - C| \geq \epsilon\} = 0$ for every $\epsilon > 0$. Hence

$$P\{X_1 \neq C\} = P\Big(\bigcup_{k=1}^{n} \{|X_1 - C| \geq k^{-1}\}\Big) \leq \sum_{k=1}^{\infty} P\{|X_1 - C| \geq k^{-1}\} = 0$$

That is, $X_1 = C$ a.s. This contradicts the non-degeneracy assumption.