Homework 2 (Chapter 2)

Exercises in Chpater 2

5.2. All we need is

$$\lim_{n\to\infty}\sum_{k=1}^{n2^n}\frac{k-1}{2^n}P\Big\{\frac{k-1}{2^n}<X\le\frac{k}{2^n}\Big\}=\sup_{0\le Y\le X}\Big\{EY;\quad Y \text{ is simple}\Big\}$$

Set

$$X_n=\sum_{k=1}^{n2^n}\frac{k-1}{2^n}1\Big\{\frac{k-1}{2^n}<X\le\frac{k}{2^n}\Big\}$$

By the fact that X_n is simple and $X_n\le X$,

$$\sup_{0\le Y\le X}\Big\{EY;\quad Y \text{ is simple}\Big\}\ge EX_n=\sum_{k=1}^{n2^n}\frac{k-1}{2^n}P\Big\{\frac{k-1}{2^n}<X\le\frac{k}{2^n}\Big\}$$

for any $n\ge 1$. Hence

$$\sup_{0\le Y\le X}\Big\{EY;\quad Y \text{ is simple}\Big\}\ge\lim_{n\to\infty}\sum_{k=1}^{n2^n}\frac{k-1}{2^n}P\Big\{\frac{k-1}{2^n}<X\le\frac{k}{2^n}\Big\}$$

On the other hand, let Y be a simple random variable with $0\le Y\le X$.. All we need to prove is that (why is this enough?)

$$\lim_{n\to\infty}EX_n\ge EY$$

Indeed, given $\epsilon>0$,

$$\begin{aligned}EX_n&\ge EX_n1_{\{X_n\ge Y-\epsilon\}}\ge E(Y-\epsilon)1_{\{X_n\ge Y-\epsilon\}}\\&\ge-\epsilon+EY1_{\{X_n\ge Y-\epsilon\}}\end{aligned}$$

By the fact $X_n\uparrow X$ we have that $Y1_{\{X_n\ge Y-\epsilon\}}\uparrow Y$ as $n\to\infty$. By monotonic convergence,

$$\lim_{n\to\infty}EY1_{\{X_n\ge Y-\epsilon\}}=EY$$

In summary,

$$\lim_{n\to\infty}EX_n\ge-\epsilon+EY$$

Letting $\epsilon\to 0^+$ ends the proof.

12.3(a) Notice $g(x)=\log^+\max\{1,x\}$ is non-negative on $[0,\infty]$. By Theorem 12.1

$$E\log^+X=\int_0^\infty P\{\log^+X>x\}dx=\int_1^\infty P\{\log X>x\}dx=\int_1^\infty P\{X>e^x\}dx$$

By variable substitution $u = e^x$

$$E\log^+ X = \int_e^\infty \frac{1}{u} P\{X > u\} du$$

By integral test, the right hand side is finite if and only if

$$\sum_{n=1}^\infty \frac{1}{n} P\{X \ge n\} < \infty$$

Problems in Chpater 2

13. **Computation of expectaton**. A key step is to re-written m_n^2 as

$$m_n^2 = \frac{1}{n}\sum_{k=1}^n \big((X_k-\mu)-(\bar{X}_n-\mu)\big)^2$$

$$= \frac{1}{n}\sum_{k=1}^n \Big\{(X_k-\mu)^2+(\bar{X}_n-\mu)^2-2(\bar{X}_n-\mu)(X_k-\mu)\Big\}$$

$$= \frac{1}{n}\sum_{k=1}^n (X_k-\mu)^2+(\bar{X}_n-\mu)^2-2(\bar{X}_n-\mu)^2 = \frac{1}{n}\sum_{k=1}^n (X_k-\mu)^2-(\bar{X}_n-\mu)^2$$

Hence

$$Em_n^2 = \frac{1}{n}\sum_{k=1}^n E(X_k-\mu)^2 - E(\bar{X}_n-\mu)^2$$

$$= \sigma^2 - \frac{1}{n^2} Var\Big(\sum_{k=1}^n X_k\Big) = \sigma^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2$$

Computation of variance (optional). Set

$$\xi_k = (X_k-\mu)^2 - E(X_k-\mu)^2 = (X_k-\mu)^2-\sigma^2$$

By the representation we obtained

$$Var(m_n^2) = Var\bigg(\frac{1}{n}\sum_{k=1}^n \xi_k - (\bar{X}_n-\mu)^2\bigg)$$

$$= Var\bigg(\frac{1}{n}\sum_{k=1}^n \xi_k\bigg) - 2E\bigg(\frac{1}{n}\sum_{k=1}^n \xi_k\bigg)\Big((\bar{X}_n-\mu)^2 - E(\bar{X}_n-\mu)^2\Big) + Var\Big((\bar{X}_n-\mu)^2\Big)$$

$$= \frac{1}{n}Var(\xi_1)^2 - \frac{2}{n}E\bigg(\sum_{k=1}^n \xi_k\bigg)(\bar{X}_n-\mu)^2 + \Big\{E(\bar{X}_n-\mu)^4 - \Big(E(\bar{X}_n-\mu)^2\Big)^2\Big\}$$

Here and elsewhere, we make use several times of the identities

$$Var(Y) = Var(Y+C) \quad \text{and} \quad Var(Y) = EY^2-(EY)^2$$

For the first term,

$$Var(\xi_1)^2 = Var\big((X-\mu)^2\big) = E(X-\mu)^4 - \big(E(X-\mu)^2\big)^2 = \mu_4 - \sigma^4$$

For the third term,

$$E(\bar{X}_n - \mu)^2 = \frac{1}{n}\sigma^2$$

and

$$\begin{aligned} E(\bar{X}_n - \mu)^4 &= \frac{1}{n^4} E\Big(\sum_{k=1}^n (X_k - \mu)\Big)^4 \\ &= \frac{1}{n^4} \sum_{i,j,k.l=1}^n E(X_i-\mu)(X_j-\mu)(X_k-\mu)(X_l-\mu) \\ &= \frac{1}{n^4}\Big\{ nE(X-\mu)^4 + \binom{4}{2}\binom{n}{2} E(X-\mu)^2 \cdot E(X-\mu)^2 \Big\} \\ &= \frac{\mu_4}{n^3} + \frac{3(n-1)}{n^3}\sigma^4 \end{aligned}$$

Therefore,

$$E(\bar{X}_n - \mu)^4 - \Big(E(\bar{X}_n-\mu)^2\Big)^2 = \frac{\mu_4}{n^3} + \frac{2n-3}{n^3}\sigma^4$$

As for the second term

$$E\Big(\sum_{k=1}^n \xi_k\Big)(\bar{X}_n - \mu)^2 = \frac{1}{n^2} E\Big(\sum_{k=1}^n \xi_k\Big)\Big(\sum_{j=1}^n (X_j - \mu)\Big)^2$$

$$= \frac{1}{n^2}\sum_{i,j,k=1}^n E\xi_k (X_i-\mu)(X_j-\mu) = \frac{1}{n} E\xi_1 (X_1-\mu)^2 = \frac{1}{n}(\mu_4 - \sigma^4)$$

In summary,

$$\begin{aligned} Var(m_n^2) &= \frac{\mu_4 - \sigma^4}{n} - \frac{2}{n^2}(\mu_4 - \sigma^4) + \Big(\frac{\mu_4}{n^3} + \frac{2n-3}{n^3}\sigma^4\Big) \\ &= \frac{\mu_4 - \sigma^4}{n} - \frac{2\mu_4 - 4\sigma^4}{n^2} + \frac{\mu_4 - 3\sigma^4}{n^3} \end{aligned}$$

as wished.

15. By Fubini's theorem,

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big(P\{X < x \le Y\} - P\{Y < x \le X\}\Big) dx \\ &= \int_{-\infty}^{\infty} \Big(P\{X < x \le Y,\ X < Y\} - P\{Y < x \le X,\ Y < X\}\Big) dx \\ &= \int_{-\infty}^{\infty} E1_{\{X<x\le Y\}} 1_{\{X<Y\}} dx - \int_{-\infty}^{\infty} E1_{\{X<x\le Y\}} 1_{\{X<Y\}} dx \\ &= E\int_{-\infty}^{\infty} 1_{\{X<x\le Y\}} 1_{\{X<Y\}} dx - E\int_{-\infty}^{\infty} 1_{\{X<x\le Y\}} 1_{\{X<Y\}} dx \end{aligned}$$

Notice that

$$\int_{-\infty}^{\infty} 1_{\{X<x\le Y\}}1_{\{X<Y\}}dx = 1_{\{X<Y\}}\int_X^Y dx = 1_{\{X<Y\}}(Y-X)$$

Similarly,

$$\int_{-\infty}^{\infty} 1_{\{X<x\le Y\}}1_{\{X<Y\}}dx = 1_{\{X>Y\}}(X-Y) = -1_{\{X>Y\}}(Y-X)$$

In summary,

$$\int_{-\infty}^{\infty}\Big(P\{X<x\le Y\} - P\{Y<x\le X\}\Big)dx$$
$$= E1_{\{X<Y\}}(Y-X) + E1_{\{X>Y\}}(Y-X) = E(Y-X) = EY - EX$$

19. First,

$$\lim_{n\to\infty} n\frac{1}{X(\omega)}1_{\{X(\omega)>n\}} = 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{1}{n}\frac{1}{X(\omega)}1_{\{X(\omega)>\frac{1}{n}\}} = 0$$

for every $\omega\in\Omega$. Indeed, the first claim follow from $1_{\{X>n\}}=0$ as $n>X(\omega)$. As for the second claim, given ω, if $X(\omega)=0$ then $1_{\{X(\omega)>\frac{1}{n}\}}=0$ and the sequence is constantly zero. When $X(\omega)>0$,

$$0\le\frac{1}{n}\frac{1}{X(\omega)}1_{\{X(\omega)>\frac{1}{n}\}}\le\frac{1}{n}\frac{1}{X(\omega)}\longrightarrow 0\qquad (n\to\infty)$$

Second, notice that

$$0\le n\frac{1}{X(\omega)}1_{\{X(\omega)>n\}}\le 1 \quad\text{and}\quad 0\le\frac{1}{n}\frac{1}{X(\omega)}1_{\{X(\omega)>\frac{1}{n}\}}\le 1$$

So the conclusion follows from the dominated convergence with dominating random variable $Y\equiv 1$.

20. By Borel-Cantelli lemma, $P\{A_n i.o.\}=1$ if and only if

$$\sum_{n=1}^{\infty}P(A_n)=\infty \tag{1}$$

Therefore all we need is to show that

$$P\Big(\bigcup_{n=1}^{\infty}A_n\Big)=1 \tag{2}$$

if and only if (1) holds.

By Morgan's law, and independence,

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = 1 - P\Big(\bigcap_{n=1}^{\infty} A_n^c\Big) = 1 - \prod_{n=1}^{\infty} P(A_n^c) = 1 - \prod_{n=1}^{\infty} \big(1 - P(A_n)\big)$$

Finally, the conclusion follows from the fact that with $P(A_n) < 1$,

$$\prod_{n=1}^{\infty} \big(1 - P(A_n)\big) = 0$$

if and only if (1) holds.

When $P(A_N) = 1$ for some $N \geq 1$, then

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) \geq P(A_N) = 1$$

It is possible that $P\{A_n \ i.o.\} = 0$ at the same time. Indeed, the "$P(A_N) = 1$ for some N" and

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

may co-exist. By Borel-Cantelly lemma,

$$P\{A_n \ i.o.\} = 0$$