Homework 2 (Chapter 2)

Exercises in Chpater 2

1.3. Need to show $\sigma(X) = \sigma\big(X^{-1}((-\infty, x]); \quad x \in \mathbf{R}\big)$ The direction of

$$\sigma(X) \supset \sigma\big(X^{-1}((-\infty, x]); \quad x \in \mathbf{R}\big)$$

is obvious. Need to show the opposite

$$\sigma(X) \subset \sigma\big(X^{-1}((-\infty, x]); \quad x \in \mathbf{R}\big)$$

Define

$$\mathcal{E} = \Big\{B \subset \mathbf{R}; \quad X^{-1}(B) \in \sigma\big(X^{-1}((-\infty, x]); \ x \in \mathbf{R}\big)\Big\}$$

All we need is

$$\mathcal{R} \subset \mathcal{E}$$

Indeed,

$$\big\{(-\infty, x]; \quad x \in \mathbf{R}\big\} \subset \mathcal{E}$$

Next, one can prove that $\mathcal{E}$ is a σ-algebra on $\mathbf{R}$. Hence

$$\mathcal{R} = \sigma\big\{(-\infty, x]; \quad x \in \mathbf{R}\big\} \subset \mathcal{E}$$

Problems in 2.20

2. We fist prove that

$$\lim_{n\to\infty} P\{|X| \geq n\} = 0$$

Indeed,

$$\bigcap_{n=1}^{\infty} \{|X| \geq n\} = \{|X| = \infty\}$$

Hence

$$P\Big(\bigcap_{n=1}^{\infty} \{|X| \geq n\}\Big) = P\{|X| = \infty\} = 0$$

By continuity theorem

$$\lim_{n\to\infty} P\{|X| \geq n\} = P\Big(\bigcap_{n=1}^{\infty} \{|X| \geq n\}\Big) = 0$$

Therefore, there is a integer $N \geq 1$ such that

$$P\{|X| \geq N\} < \epsilon$$

Set

$$X_\epsilon = X1_{\{|X|<N\}}$$

We have $|X_\epsilon| \le N$ and

$$P\{X \ne X_\epsilon\} \le P\{|X| \ge N\} < \epsilon$$

4. (a) Since $F(x) - F(x-0) = P\{X = x\}$ for any $x \in \mathbf{R}$. $P\{X = x\} = 0$ whenever $x \notin J_f$. In addition

$$P\{x - h < X \le x + h\} = F(x+h) - F(x-h).$$

Therefore, all we need is to show that for any decresing sequence h_n with $h_n \to 0$ $(n \to \infty)$

$$\lim_{n\to\infty} P\{x - h_n < X \le x + h_n\} = P\{X = x\} \qquad \forall x \in \mathbf{R}$$

Indeed,

$$\bigcap_{n=1}^{\infty} \{x - h_n < X \le x + h_n\} = \{X = x\}$$

and $\{x - h_n < X \le x + h_n\}$ is a non-increasing sequence. By continuity theorem,

$$\lim_{n\to\infty} P\{x - h_n < X \le x + h_n\} = P\Big(\bigcap_{n=1}^{\infty} \{x - h_n < X \le x + h_n\}\Big) = P\{X = x\}$$

5 .Notice that X takes integer values.

$$P(n < X \le n+m) = \sum_{k=n+1}^{n+m} P(X = k)$$

Therefore, by Fubini's theorem

$$\sum_{n=-\infty}^{\infty} P(n < X \le n+m) = \sum_{n=-\infty}^{\infty} \sum_{k=n+1}^{n+m} P(X = k) = \sum_{k=-\infty}^{\infty} \sum_{n=k-m}^{k-1} P(X = k)$$

$$= \sum_{k=-\infty}^{\infty} P(X = k) \sum_{n=k-m}^{k-1} 1 = m \sum_{k=-\infty}^{\infty} P(X = k) == m$$

8. Assume that

$$\sum_{n=1}^{\infty} P(X_n > A) = \infty \qquad \forall A > 0$$

By Problem 12, p.24,

$$P\Big(\sup_{n\ge 1} X_n > A\Big) = P\Big(\bigcup_{n=1}^{\infty} \{X_n > A\}\Big) = 1 \qquad \forall A > 0$$

Therefore,

$$P\Big(\sup_{n\geq 1} X_n = \infty\Big) = P\Big(\bigcap_{k=1}^{\infty}\{\sup_{n\geq 1} X_n > k\}\Big) = 1 - P\Big(\bigcup_{k=1}^{\infty}\{\sup_{n\geq 1} X_n \leq k\}\Big)$$

By sub-additivity,

$$P\Big(\bigcup_{k=1}^{\infty}\{\sup_{n\geq 1} X_n \leq k\}\Big) \leq \sum_{k=1}^{\infty} P\Big\{\sup_{n\geq 1} X_n \leq k\Big\} = 0$$

We have

$$\sup_{n\geq 1} X_n = \infty \qquad a.s.$$

Assume that

$$\sum_{n=1}^{\infty} P(X_n > A) < \infty \qquad \text{for some } A > 0$$

To prove that

$$\sup_{n\geq 1} X_n < \infty \qquad a.s.$$

All we need is to show

$$\limsup_{n\to\infty} X_n < \infty \qquad a.s.$$

Notice that for any $A > 0$,

$$\Big\{\limsup_{n\to\infty} X_n > A\Big\} = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\{X_m > A\}$$

By continuity theorem,

$$P\Big\{\limsup_{n\to\infty} X_n > A\Big\} = P\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\{X_m > A\}\Big) = \lim_{n\to\infty} P\Big(\bigcup_{m=n}^{\infty}\{X_m > A\}\Big)$$

By sub-additivity

$$P\Big(\bigcup_{m=n}^{\infty}\{X_m > A\}\Big) \leq \sum_{m=n}^{\infty} P\{X_m > A\} \longrightarrow 0 \qquad (n\to\infty)$$

for any $A > 0$ that make the series converge. In summary, there is a constant $A > 0$ such that

$$P\Big\{\limsup_{n\to\infty} X_n > A\Big\} = 0$$

Or

$$\limsup_{n\to\infty} X_n \leq A < \infty \qquad a.s.$$