Homework 1

Problems in 1.6.

4. We need to establish the relations

$$\sigma\big(\{A_1,\cdots,A_n\}\big)\subset\sigma\big(\{B_1,\cdots,B_n\}\big)\quad\text{and}\quad\sigma\big(\{A_1,\cdots,A_n\}\big)\supset\sigma\big(\{B_1,\cdots,B_n\}\big)$$

Set $A_0=\phi$. By the fact the $A_1,\cdots,A_n$ are disjoint, $A_k=B_k\setminus B_{k-1}\in\sigma\big(\{B_1,\cdots,B_n\}\big)$ $(k=1,\cdots n)$. Or, $\{A_1,\cdots,A_n\}\subset\sigma\big(\{B_1,\cdots,B_n\}\big)$. Since $\sigma\big(\{A_1,\cdots,A_n\}\big)$ is the smallest σ-algebra that takes $\{A_1,\cdots,A_n\}$ as its sub-class. So we must have

$$\sigma\big(\{A_1,\cdots,A_n\}\big)\subset\sigma\big(\{B_1,\cdots,B_n\}\big)$$

Another direction can be established in a similar way.

5. (a). Indeed, we can prove that for any $\epsilon>0$ there is a subsequence $\{n_k\}$ such that

$$P\Big(\bigcap_{k=1}^\infty A_{n_k}\Big)\ge 1-\epsilon$$

By assumption, for each $k\ge1$, there is n_k such that

$$P(A_{n_k})\ge 1-\frac{1}{2^k}\epsilon$$

Clearly, we can make $n_k<n_{k+1}$ for any $k\ge1$. By Morgan's law,

$$P\Big(\Big\{\bigcap_{k=1}^\infty A_{n_k}\Big\}^c\Big)=P\Big(\bigcup_{k=1}^\infty A_{n_k}^c\Big)\le\sum_{k=1}^\infty P(A_{n_k})\le\sum_{k=1}^\infty\frac{1}{2^k}\epsilon=\epsilon$$

(b). Assume that $\{A_n\}\subset\mathcal{F}$ be an independent sequence with $P(A_n)=\alpha>0$ (you may think A_n is the event of "success in game n" in a Bernoulli trial). Then for any n_k,

$$P\Big(\bigcap_{k=1}^\infty A_{n_k}\Big)=\prod_{k=1}^\infty P(A_{n_k})=0$$

(c). By continuity theorem (Theorem 3.1, p.11)

$$P\Big(\bigcap_{n=1}^\infty A_n\Big)=\lim_{n\to\infty}P(A_n)\ge\alpha$$

(d). By Morgan's law

$$P\Big(\bigcup_{n=1}^\infty A_n\Big)=1-P\Big(\bigcap_{n=1}^\infty A_n^c\Big)$$

Notice A_n^c is non-increasing with

$$P(A_n^c) = 1 - P(A_n) \geq 1 - \alpha$$

Applying the conclusion of Part (c) to the sequence $\{A_n^c\}$,

$$P\Big(\bigcap_{n=1}^{\infty} A_n^c\Big) \geq 1 - \alpha$$

So we have

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \alpha$$

11. By Morgan's law and independence,

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = 1 - P\Big(\Big\{\bigcup_{k=1}^{n} A_k\Big\}^c\Big) = 1 - P\Big(\bigcap_{k=1}^{n} A_k^c\Big) = 1 - \prod_{k=1}^{n} P(A_k^c) = 1 - \prod_{k=1}^{n} \big(1 - P(A_k)\big)$$

12. By the inequality $e^{-x} \geq 1 - x$ $(x > 0)$ and the identity from Problem 11,

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = 1 - \prod_{k=1}^{n} \big(1 - P(A_k)\big) \geq 1 - \prod_{k=1}^{n} \exp\big\{-P(A_k)\big\} = 1 - \exp\Big\{-\sum_{k=1}^{n} P(A_k)\Big\}$$

Assume that

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

By continuity theorem

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = P\Big(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} A_k\Big) = \lim_{n\to\infty} P\Big(\bigcup_{k=1}^{n} A_k\Big) \geq 1 - \lim_{n\to\infty} \exp\Big\{-\sum_{k=1}^{n} P(A_k)\Big\} = 1$$

13. The problem is to prove

$$\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m \neq \phi$$

All we need is to show

$$\lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m\Big) \geq \eta > 0$$

Indeed, by continuity

$$\lambda\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m\Big) = \lim_{n\to\infty} \lambda\Big(\bigcup_{m=n}^{\infty} A_m\Big)$$

So the conlusion follows from the relation

$$\lambda\Big(\bigcup_{m=n}^{\infty} A_m\Big) \geq \lambda(A_n) \geq \eta$$