Do all problems and give the process of your solution.

1. A Markov chain \( \{X_0, X_1, X_2, \cdots, \} \) has the transition probability

\[
\begin{pmatrix}
0 & 1 & 2 \\
0 & 0.1 & 0.8 \\
1 & 0.2 & 0.6 \\
2 & 0.3 & 0.4
\end{pmatrix}
\]

Determine the conditional probabilities

(1). \( P[X_1 = 1, X_2 = 1|X_0 = 0] \)

Solution. 

\[
P[X_1 = 1, X_2 = 1|X_0 = 0] = P_{0,1}P_{1,1} = 0.1 \times 0.2 = 0.02
\]

(2). \( P[X_2 = 1|X_0 = 0] \)

Solution. 

\[
P[X_2 = 1|X_0 = 0] = P[X_1 = 0, X_2 = 1|X_0 = 0] \\
+ P[X_1 = 1, X_2 = 1|X_0 = 0] \\
+ P[X_1 = 2, X_2 = 1|X_0 = 0] \\
= P_{0,0}P_{0,1} + P_{0,1}P_{1,1} + P_{0,2}P_{2,1} \\
= 0.1 \times 0.1 + 0.1 \times 0.2 + 0.8 \times 0.3 = 0.27
\]

2. Consider the Markov chain \( \{X_n\} \) with transition probability matrix

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 \\
1 & q & 0 & p & 0 \\
2 & 0 & q & 0 & p \\
3 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( 0 < p < 1 \) and \( q = 1 - p \). Let \( T = \min \{n \geq 0; X_n \in \{0,3\}\} \) be the first time the process is absorbed. Find (in terms of \( p \) and \( q \))

(1). \( P[X_T = 0|X_0 = 1] \) and \( P[X_T = 0|X_0 = 2] \)

Solution. Write \( u_1 = P[X_T = 0|X_0 = 1] \) and \( u_1 = P[X_T = 0|X_0 = 2] \). We have that

\[
\begin{cases}
u_1 - pu_2 = q \\
-(q + p)u_1 + u_2 = 0
\end{cases}
\]

So

\[
\begin{cases}
u_1 = \frac{q}{1 - pq} \\
u_2 = \frac{q^2}{1 - pq}
\end{cases}
\]

(2). \( E[T|X_0 = 1] \) and \( E[T|X_0 = 2] \)

Solution. Write \( v_1 = E[T|X_0 = 1] \) and \( v_1 = E[T|X_0 = 2] \). We have that

\[
\begin{cases}
v_1 - pu_2 = 1 \\
-(q + p)v_1 + v_2 = 1
\end{cases}
\]

So

\[
\begin{cases}
v_1 = \frac{1 + p}{1 - pq} \\
v_2 = \frac{1 + q}{1 - pq}
\end{cases}
\]
3. A coin is tossed repeatedly until four heads in a row appear. Let $X_n$ record the current number of successive heads that have appeared. That is, $X_n = 0$ if the $n$th toss resulted in tail; $X_n = 1$ if the $n$th toss is head and the $(n-1)$th toss was tail; and so on. Write out the transition probability matrix with minimal dimension.

**Solution.** The transition probability matrix is

$$
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
3 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
4 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$