Lemma. For all positive integers \( n \),

\[
(-1)^k \binom{n}{k} = 0.
\]

Proof. Let \( \mathcal{E} \) denote the set of all subsets of \([n]\) having even cardinality, and \( \mathcal{O} \) the set of all subsets of \([n]\) having odd cardinality. Formula (1) is equivalent to the assertion that \( |\mathcal{E}| = |\mathcal{O}| \). It remains only to observe that the map from \( \mathcal{E} \) to \( \mathcal{O} \) defined by (i) \( E \mapsto E - \{1\} \) if \( 1 \in E \), and (ii) \( E \mapsto E \cup \{1\} \) if \( 1 \notin E \) is a bijection.

The Characteristic Function of a Set. Suppose that \( A \) and \( B \) are sets and \( B \subset A \). The characteristic function of \( B \), denoted \( \chi_B \), is defined for all \( a \in A \) by (i) \( \chi_B (a) = 1 \) if \( a \in B \), and (ii) \( \chi_B (a) = 0 \) if \( a \notin B \). Note that if \( B \) is finite, then

\[
|B| = \sum_{a \in A} \chi_B (a).
\]

Theorem (Principle of Inclusion and Exclusion, a.k.a. the Sieve Formula). Let \( A_1, \ldots, A_n \) be a sequence of subsets of the finite set \( A \). Then

\[
|A_1 \cup \ldots \cup A_n| = \sum_{\emptyset \neq J \subset [n]} (-1)^{|J|-1} |\bigcap_{i \in J} A_i|.
\]

Proof. Let \( A_{\emptyset} := \bigcap_{i \in J} A_i \). By (2), formula (3) is equivalent to

\[
\sum_{a \in A} \chi_{A_1 \cup \ldots \cup A_n} (a) = \sum_{\emptyset \neq J \subset [n]} (-1)^{|J|-1} \sum_{a \in A} \chi_{A_i} (a) = \sum_{a \in A} \sum_{\emptyset \neq J \subset [n]} (-1)^{|J|-1} \chi_{A_i} (a).
\]

And formula (4) holds if, for each \( a \in A \),

\[
\chi_{A_1 \cup \ldots \cup A_n} (a) = \sum_{\emptyset \neq J \subset [n]} (-1)^{|J|-1} \chi_{A_i} (a).
\]

If \( a \) is an element of none of the sets \( A_i \), then (5) holds in the form \( 0 = 0 \). Suppose then that \( a \in A_i \) for precisely those \( i \in J \), where \( |J| = j > 0 \). Then the left-hand side of (5) is equal to 1, and the right-hand side of (5) is equal to

\[
\sum_{\emptyset \neq J \subset [j]} (-1)^{|J|-1} \binom{j}{i} = 1,
\]

by the Lemma.