Logic and Proof III Carl Wagner

Some Paradoxes of Propositional Logic

First, let's recall the correct meaning of the word *paradox*, which is incorrectly used by many people as a synonym for the word *contradiction.* A paradox is an *apparent contradiction*, i.e., an assertion that seems contradictory, but is not. There are a number of paradoxes in logic and set theory, and even more in probability theory, and it is always a worthwhile exercise to try to explain, and thereby dispel, such paradoxes.

1. A paradoxical equivalence

Here is a well-known example of a paradoxical equivalence of propositional logic, which you can easily verify by truth tables:

(1) $$(M \wedge B) \rightarrow C \;\Leftrightarrow\; (M \rightarrow C) \vee (B \rightarrow C).$$

Now consider a special case of (1). It is a well known theorem of first year calculus that if a sequence of real numbers is monotone and bounded, then it is convergent. But neither monotonicity alone nor boundedness alone is sufficient to guarantee convergence. Let s be a sequence of real numbers (i.e., a function from the set of nonnegative integers to the set of real numbers), with M(s) asserting that s is monotone, B(s) asserting that s is bounded, and C(s) asserting that s converges. The aforementioned theorem tells us that the proposition

(2) $$(M(s) \wedge B(s) \rightarrow C(s)$$

is true for all s. But by (1), it follows (in fact, just from the $\Rightarrow$ part of $\Leftrightarrow$) that

(3) $$(M(s) \rightarrow C(s) \vee (B(s) \rightarrow C(s)$$

is true for all s, from which it follows that the following proposition (which is simply either true or false,, since it contains no free variables) is true:

(4) $$\forall\, s\, [(M(s) \rightarrow C(s) \vee (B(s) \rightarrow C(s)].$$

People often find (4) to be puzzling, thinking that it asserts the (false) proposition that *either monotonicity guarantees convergence or boundedness guarantees convergence*. But that is not what (4) – which is in fact true – says. The aforementioned italicized statement is properly represented as

(5) $$\forall\, s\, [(M(s) \rightarrow C(s)] \;\vee\; \forall\, s\, [(B(s) \rightarrow C(s)],$$

and (5) does not follow from (4), as the very example that we have been considering shows.

The mistake that is made in passing from (4) to (5) is clearly *the incorrect assumption that universal quantification may be distributed through disjunctions.* But while universal quantification does distribute through conjunctions, i.e.,

(6) $\forall x\,(P(x) \wedge Q(x))$ is equivalent to $\forall x\, P(x) \;\wedge\; \forall x\, Q(x)$,

it does not distribute through disjunctions.(On the other hand, existential quantification distributes through disjunctions, but not conjunctions, as you may easily check).

2. **A paradoxical implication**

One can easily confirm using truth tables the following implication of propositional logic:

(7) $(C \to R) \wedge (P \to D) \;\Rightarrow\; (C \to D) \vee (P \to R).$

Now let C(x) assert that x belongs to a conservative political club, R(x) that x votes Republican, P(x) that x belongs to a progressive political club, and D(x) that x votes Democratic. Let us suppose that

(8) $(C(x) \to R(x) \wedge (P(x) \to D(x))$

is true for all x in some population U. It then follows from (7) that

(9) $(C(x) \to D(x)) \vee (P(x) \to R(x))$

is true for all x, from which it follows that the following proposition (which is simply either true or false,, since it contains no free variables) is true:

(10) $\forall x\,[(C(x) \to D(x)) \vee (P(x) \to R(x))].$

People often find (10) puzzling, thinking that it asserts the proposition (which contradicts our assumption) that *it is either the case that everyone who belongs to a conservative political club votes Democratic, or everyone who belongs to a progressive political club votes Republican*. But that is not what (10) – which is completely consistent with the assumption that (8) is true for all x – says. The aforementioned italicized statement is properly expressed as

(11) $\forall x(C(x) \to D(x)) \vee \forall x(P(x) \to R(x))$,

and (11) does not follow from (10), as the very example that we have been considering shows. Once again, the error lies in the mistaken idea that universal quantification may be distributed through disjunctions.

3. A paradoxical tautology

One may easily verify by a truth table that the following wff of propositional logic is a tautology:

(12) $$(I \to C) \vee (C \to D).$$

Now consider the universe U consisting of all real valued functions f defined on the interval [0,1]. Let I(f) assert that f is Riemann integrable, let C(f) assert that f is continuous, and let D(f) assert that f is differentiable. It is well known that there are Riemann integrable functions that are not continuous, and that there are continuous functions that are not differentiable. On the other hand, it follows from (12) that the proposition

(13) $$(I(f) \to C(f)) \vee (C(f) \to D(f))$$

is true for every f, from which it follows that the following proposition (which is simply true or false, since it contains no free variables) is true:

(14) $$\forall\, f\, [(I(f) \to C(f)) \vee (C(f) \to D(f))].$$

By now, you should be able to see where this is going, and so I leave the elaboration of this example as an exercise.

The moral of the foregoing stories is:

If you think that universal quantification distributes over disjunction, then you will conclude that propositional logic is "a tale told by an idiot,…, signifying nothing."