

## Time-stepping schemes - higher order

$$U(x_i, t) \approx U_i(t)$$

1. Method of lines: Discretize only in space, leaving time (derivative) continuous.

Then the PDE reduces to a system of ODEs in time.  $\frac{d}{dt} U_i(t) = FDE(U_i(t))$   
 $i=1, \dots, M$

and one can use an industrial strength ODE solver for the time stepping,

(LSODE or VODE), RKsuite, GSL-ode, etc., ...

This makes sense when using a high order spatial discretization, such as WENO.

But then one must use a TVD time-stepping scheme! else oscillations may occur!

best are SSP schemes (Strong Stability Preserving) of Gottlieb-Shu;  $\|U^{n+1}\| \leq \|U^n\|$  for, under some CFL condition

2. Runge-Kutta TVD schemes have been constructed, of orders 2, 3, 4, 5, by Shu, etc.

3. Also multistep TVD schemes have been found, up to 4<sup>th</sup> order, by Shu.

Both RK and multistep require positive weights to be TVD, possible only up to 3<sup>rd</sup> order.

The time step (CFL condition) is smaller for such high order schemes than for 1<sup>st</sup> order Euler.

But there are schemes with same CFL:

Shu-Osher 3<sup>rd</sup> order TVD-RK for  $\vec{U}_t = g(\vec{U})$ : which he has been using in WENO, LDG, ... (1988)

$$w_1 = U^n + \Delta t \cdot g(U^n) \quad (\text{Forward Euler})$$

$$w_2 = \frac{3}{4} U^n + \frac{1}{4} [w_1 + \Delta t \cdot g(w_1)]$$

$$U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} [w_2 + \Delta t \cdot g(w_2)]$$

uses 3 evaluations of  $g(\cdot)$  and achieves 3<sup>rd</sup> order in time.

Under the same CFL as Forward Euler it preserves Max Principle

(since it is a convex combination of Euler steps!) and monotone, and SSP!

The limiter should be applied at each stage of the RK.

CFL for SSP schemes: If Forward Euler is stable in some norm or seminorm under a CFL:  $\Delta t \leq \Delta t_{\text{expt}}$

then the high order SSP is also stable (in same norm/seminorm) under the CFL

$$\Delta t \leq C \cdot \Delta t_{\text{expt}}, \text{ for some } C.$$

Shu-Osher needs  $C=1$ !

Secret of being SSP: scheme can be written as convex combination of Euler steps!

Max-Principle-satisfying scheme of Zhang-Shu 2010

Zhang-Shu: MaxPrinc satisfies and positivity preserving  
high order---: survey, new developments  
Proc. Royal Soc. A, 2011

best of all!

for FV and RKDG on cell averages

Max Principle-satisfying schemes: if  $U_i^n, i=1, \dots, M$  are in  $[m, M]$  then so are  $\bar{U}_i^{n+1}$

desirable for scalar  $u_t + \nabla \cdot F(u) = 0$

But systems of conserv. laws do not satisfy Max Principle, best we'd like is positivity preserving.

The Zhang-Shu scheme preserves positivity for Euler equations too!

and extended to 2D triangular meshes and rectangular. Applies also to RKDG!

$$\text{Conservative schemes: } U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} [h(\bar{u}_{i-\frac{1}{2}}, u_{i+\frac{1}{2}}^+) - h(\bar{u}_{i+\frac{1}{2}}, u_{i-\frac{1}{2}}^-)] \quad (1)$$

with  $h(a, b)$  a monotone flux [Lip., consistent,  $h(\uparrow, \downarrow)$ , e.g. Lax-Friedrichs, Godunov]

$u_{i \pm \frac{1}{2}}^\pm$  are reconstructed face values from high order polynomials that preserve averages.

For Max Principle-satisfying want: if  $m \leq U_i^n, u_{i \pm \frac{1}{2}}^\pm \leq M$  then  $\bar{U}_i^{n+1} \in [m, M]$

Some flux/slope limiters can do it, but they kill accuracy near smooth extrema!

(1) satisfies it for CFL:  $a \frac{\Delta t}{\Delta x} \leq 1$ ,  $a = \max |F'(u)|$ .

New approach! Consider an  $N$ -point Gauss-Lobatto quadrature on  $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ ,  $\int_N^N f(x) dx = \sum_{\alpha=1}^N w_\alpha f(\tilde{x}_\alpha)$

for high order which is exact for polynomials of degree  $\leq 2N-3$ .

Quadrature pts  $S_i = \{x_{i-\frac{1}{2}} = \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{N-1}, \tilde{x}_N = x_{i+\frac{1}{2}}\}$  in  $I_i$

Weights  $\hat{w}_\alpha$  for interval  $[-\frac{1}{2}, \frac{1}{2}]$ ,  $\sum_{\alpha=1}^N \hat{w}_\alpha = 1$ , ( $\hat{w}_1 = \hat{w}_N$ ) for Gauss-Legendre quadrature

Choose  $N$  as smallest:  $2N-3 \geq k$  = desired order of approximation, construct interpolant  $p_i(x)$  of deg  $k$  for  $S_i$  that preserves  $\bar{U}_i$ :

(so  $N=3$  for  $k=3$ ,  $N=4$  for  $k=5$ ,  $N=5$  for  $k=7$ ,  $N=6$  for  $k=9, \dots$ )

$$\text{Then } \bar{U}_i^n = \frac{1}{\Delta x} \int_{I_i} p_i(x) dx = \sum_{\alpha=1}^N \hat{w}_\alpha p_i(\tilde{x}_\alpha) = \hat{w}_1 u_{i-\frac{1}{2}}^+ + \sum_{\alpha=2}^{N-1} \hat{w}_\alpha p_i(\tilde{x}_\alpha) + \hat{w}_N u_{i+\frac{1}{2}}^-$$

$\parallel$                                      $\parallel$   
 $p_i(x_{i-\frac{1}{2}})$                              $p_i(x_{i+\frac{1}{2}})$

MaxPrincThm: If  $u_{i-\frac{1}{2}}^+, u_{i+\frac{1}{2}}^-, p_i(\tilde{x}_\alpha), \alpha=1, \dots, N$  are in  $[m, M]$  then  $\bar{U}_i^{n+1}$  is also in  $[m, M]$

under the CFL condition  $a \frac{\Delta t}{\Delta x} \leq \hat{w}_1$ ,  $a = \max_u |F'(u)|$ .  $\hat{w}_1 = \hat{w}_N = \frac{2}{N(N-1)}$ ?

So the scheme will satisfy MaxPrinciple if we can make  $p_i(x) \in [m, M]$  for all  $x \in S_i$

This can be achieved by modifying the interpolating polynomial  $p_i(x)$  by the Liu-Osher limiter! to  $\tilde{p}_i(x)$ :

$$w' = g(w),$$

Note: For timesteping must use a high order SSP stepper, Shu-Osher 3rd order SSP RK (has CFL coeff  $C=1$ :  $\Delta t \leq c \frac{\Delta t}{\text{exy}}$ )

$$\text{or 3rd order SSP multistep (CFL coeff } C = \frac{1}{3})! \quad \tilde{w}^{n+1} = \frac{16}{27} [w^n + 3\Delta t G(w^n)] + \frac{11}{27} [w^{n-3} + \frac{12}{11} \Delta t G(w^{n-3})]$$

They are convex combinations of Euler stepper  $\therefore$  satisfying Max Principle

Limiter must be applied at each stage or step of multistep.

modified interpolant by

$$\text{Liu-Osher limiter (1996)}: \tilde{P}_i(x) = \bar{U}_i^n + \theta \cdot (P_i(x) - \bar{U}_i^n) \quad \tilde{U}_i \in \mathbb{R}, \bar{U}_i^n \in \mathbb{R}, \theta \in [0, 1], \forall x \in \mathbb{R}$$

$$\theta = \min \left\{ \left| \frac{M_i - \bar{U}_i^n}{M_i - U_i^n} \right|, \left| \frac{m_i - \bar{U}_i^n}{m_i - U_i^n} \right|, 1 \right\}$$

with  $m_i = \min_{x \in S_i} P_i(x)$ ,  $M_i = \max_{x \in S_i} P_i(x)$ . This limiter is  $(k+l)$ -order, thus preserves the  $k$ -th order approximation of  $P_i(x)$ .

Set  $\tilde{u}_{i-\frac{1}{2}}^+ = \tilde{P}_i(x_{i-\frac{1}{2}})$ ,  $\tilde{u}_{i+\frac{1}{2}}^- = \tilde{P}_i(x_{i+\frac{1}{2}})$ ,  $i=2, \dots, M-1$

$$\text{The scheme: } \bar{U}_i^{n+1} = \bar{U}_i^n + \frac{\Delta t}{\Delta x} \left[ h(\tilde{u}_{i-\frac{1}{2}}^-, \tilde{u}_{i+\frac{1}{2}}^+) - h(\tilde{u}_{i+\frac{1}{2}}^-, \tilde{u}_{i+\frac{1}{2}}^+) \right]$$

will satisfy the Max Principle, hence also positivity preserving!

The Liu-Osher limiter is  $O(\Delta x^{k+l})$  accurate, so it preserves the  $k$ -th order approximation of  $P_i(x)$  and it does not change the cell average of  $P_i(x)$ , i.e. conservative!

∴ the <sup>new</sup> limited scheme is conservative, Max Principle-satisfying,  $k$ -th order in small  $\Delta x$  positivity preserving.

Zhang-Shen have extended it to 2D rectangular and triangular meshes!

Moreover, the scheme is L-stable: If solution is non-negative, i.e.  $u \geq 0$

$$\text{then } \sum_i |\bar{U}_i^{n+1}| = \sum_i |\bar{U}_i^n|$$

General procedure for constructing high order positivity-preserving schemes  
 for Euler system of Fluid Dynamics:  $\vec{w}_t + \vec{F}(\vec{w})_x = \vec{0}$

$$\vec{w} = \begin{bmatrix} p \\ m \\ E \end{bmatrix}, \vec{F}(\vec{w}) = \begin{bmatrix} m \\ \rho v^2 + P \\ (\rho + P)v \end{bmatrix} \quad \begin{aligned} m &= \rho v \\ E &= \frac{1}{2} \rho v^2 + \rho e(p, P) \end{aligned}$$

(for ideal gas:  $P = (\gamma - 1)\rho e$ ,  $\gamma > 1$  ratio of specific heats,  $\gamma = 1.4$  for air)

$\Rightarrow$  speed of sound  $c = \sqrt{\frac{\gamma P}{\rho}}$ , eigenvalues of Jacobian  $F'(\vec{w})$  are  $v - c, v, v + c$ )

Set of admissible states:  $G = \{\vec{w} : p > 0, P > 0\}$

Step 1: Prove  $G = \{\vec{w} : p > 0, P > 0\}$  is convex set (very crucial!)

Step 2: Prove the 1<sup>st</sup> order scheme  $\vec{w}_i^{n+1} = \vec{w}_i^n + \frac{\Delta t}{\Delta x} [h(\vec{w}_{i-1}^n, \vec{w}_i^n) - h(\vec{w}_i^n, \vec{w}_{i+1}^n)]$   
 preserves positivity

Step 3: Find a sufficient condition for Forward Euler scheme to produce  $\vec{w}_i^{n+1} \in G$   
 under some CFL condition.

Then convexity of  $G$  will imply that high order SSP (RK or multistep)  
 will preserve positivity.

Step 4: Construct a limiter to enforce the sufficient condition of Step 3.

C-W Shu and his army of students have applied this procedure to a great variety of problems:

Euler eqn of CFD with various EoS, source terms

Level-set eqn with div free  $\vec{v}$

Vlasov-Poisson eqn for distribution function

shallow water eqn.

Vlasov-Boltzmann transport eqns

...