

ENO Schemes

During 1984-8, Harten, Osher, Engquist, Shu, et al., developed the so called Essentially Non-Oscillatory schemes and ^{later} Weighted ENO schemes to overcome the limitations of TVD schemes. These are probably the best high order schemes available for very demanding shock problems. But they are rather complicated and expensive, so the search continues unabated... No convergence proofs exist (yet)!

A complete survey paper: Chi-Wang Shu, ICASE report 97-65, 1997.

Shu, SIAM Review 51(1): 82-120, 2009.

The issue is to suppress spurious oscillations that higher order FD schemes suffer from.

One way is to add artificial viscosity near discontinuities, which must be tuned for each problem.

Another way is to use flux-limiters to preserve the TVD property, which degrades to 1st order even near smooth extrema. Many of the troubles are due to interpolation needed to find face values from cell averages!

ENO satisfies the TVD property only approximately: $\text{TV}[U^{n+1}] \leq \text{TV}[U^n] + O(\Delta x^k)$, some k

It's basic ingredient is a "reconstruction" of the solution from the cell averages,

(polynomial interpolation of degree $k-1$), from which approximations to values at the cell faces are found.

The trick in the ENO reconstruction is to use a adaptive moving stencil for interpolation

instead of the standard fixed stencil, (trying to avoid the discontinuous cell in the stencil, if possible)

The rest of the steps are the same as in Godunov type methods,

Thus, the basic steps for solving $U_t + F(u)_x = 0$ are: Discretize by FV (or by FD) and

1. From the known cell averages U_i^n , follow the ENO or WENO ^{reconstruction} procedure

to obtain the k -th order reconstructed values $u_{i-\frac{1}{2}}^-, u_{i+\frac{1}{2}}^+$ $\forall i$
at cell face $i \pm \frac{1}{2}$, $\forall i$

2. Choose a monotone flux function $h(a, b)$ such that

- $h(a, b)$ is consistent with physical flux F , i.e. $h(a, a) = F(a)$
- $h(a, b)$ is nondecreasing in a and nonincreasing in b : $h(\uparrow, \downarrow)$
- $h(a, b)$ is Lipschitz in both a and b .

and compute the numerical flux $\tilde{F}_{i-\frac{1}{2}} = h(u_{i-\frac{1}{2}}^-, u_{i+\frac{1}{2}}^+) \quad \forall i$

3. Update via the discrete PDE: $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} [\tilde{F}_{i+\frac{1}{2}} - \tilde{F}_{i-\frac{1}{2}}] \quad \forall i$

Examples of monotone flux functions $h(a, b)$ are:

a. Godunov flux: $h(a, b) = \begin{cases} \min_{a \leq u \leq b} F(u) & \text{if } a \leq b \\ \max_{b \leq u \leq a} F(u) & \text{if } a > b \end{cases}$ (least dissipative)

b. Engquist-Osher flux: $h(a, b) = \int_a^b \max\{F'(u), 0\} du + \int_a^b \min\{F'(u), 0\} du + F(0)$

c. Lax-Friedrichs flux: $h(a, b) = \frac{1}{2} [F(a) + F(b) - \alpha \cdot (b-a)]$, $\alpha = \max_u \{F'(u)\}$ (=constant) (most dissipative)

Turns out that for high order reconstructions, which choice is made has little effect, so might as well choose the cheapest (Lax-Friedrichs)!

Basic steps of the ENO reconstruction: [Shu, ICASE 97-65 p. 14-15]

Given the cell averages U_i , construct a piecewise polynomial of degree $\leq k-1$ as follows:

1. In cell I_i , start with the 2-pt stencil $S_2(i) = \{x_{i-1/2}, x_{i+1/2}\}$ for the primitive $U(x) = \int_{-\infty}^x u(\xi) d\xi$.

Then $U(x_{i+1/2}) = \sum_{j=-\infty}^{i+1} \int_{x_{j-1/2}}^{x_{j+1/2}} u(\xi) d\xi = \sum_{j=-\infty}^i U_j \Delta x_j$, so the face values of $U(x)$ are known exactly.

2. Compute the divided differences of U (which measure the smoothness of U)

some smoothness indicator $U[x_{i-3/2}, x_{i+3/2}] = \frac{U(x_{i+1/2}) - U(x_{i-1/2})}{x_{i+1/2} - x_{i-1/2}} = U_i$, $U[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$, $U[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]$

4. If $|U[x_{i-3/2}, x_{i-1/2}, x_{i+1/2}]| < |U[x_{i-1/2}, x_{i+1/2}, x_{i+3/2}]|$ add $x_{i-3/2}$ to the stencil
else add $x_{i+3/2}$ to the stencil. i.e. we add the point towards which U is smoother.

3. Repeat on the 3-pt stencil using 3rd order divided differences, ..., adding left or right neighbors to the stencil, until a stencil $S_{k+1}(i)$ of k points is reached.

4. Construct the interpolating polynomial of degree $k-1$ in I_i , which approximates $u(x)$ to order k :

$$P_i(x) = u(x) + O(\Delta x^k), x \in I_i, i=1, \dots, M \quad \text{provided } u(x) \text{ is smooth in } I_i.$$

Then $U_{i-1/2}^+ := P_i(x_{i-1/2})$ and $U_{i+1/2}^- = P_i(x_{i+1/2})$ are k -th order approximations to the face values of $u(x)$, (reconstructed values). WENO uses weighted convex combinations of the $U_{i-1/2}^{(r)}$, $r=0, \dots, k$, from all k stencils.

This ENO reconstruction is TVB: $TV(P) \leq TV(u) + O(\Delta x^k)$

The secret of success of the ENO reconstruction is this moving, adaptive stencil!

Note: for uniform mesh, use undivided differences

WENO schemes for hyperbolic conservation laws [Shu 2010: High-order WENO, SIAM]

$$u_t + F(u) = 0$$

Finite Volume schemes: integrate over each control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$; set $\bar{U}_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t) dx$

$$\frac{d\bar{U}_i}{dt} = \frac{1}{\Delta x} [F(u_{i-\frac{1}{2}}) - F(u_{i+\frac{1}{2}})], \quad u_{i\pm\frac{1}{2}} = \text{face values of } u \text{ at}$$

need approximations for fluxes, which is done via

monotone numerical flux function: $f(\bar{u}, u^+)$ with the properties:

- nondecreasing in \bar{u} and nonincreasing in u^+ : $f(\uparrow, \downarrow)$
- consistent with physical flux: $f(u, u) = F(u)$
- Lipschitz in both \bar{u}, u^+

$u_{i\pm\frac{1}{2}}^\pm$ are "reconstructions" via WENO, described below.

Examples of monotone flux functions:

1. Godunov flux: $f^{\text{God}}(\bar{u}, u^+) = \begin{cases} \min\{f(u)\} & \text{if } \bar{u} \leq u^+ \\ \max\{f(u)\} & \text{if } \bar{u} > u^+ \end{cases}$
2. Lax-Friedrichs flux: $f^{\text{LF}}(\bar{u}, u^+) = \frac{1}{2} [F(\bar{u}) + F(u^+) - \alpha \cdot (u^+ - \bar{u})], \quad \alpha = \max_u |F'(u)|$

3. Engquist-Osher flux: $f^{\text{EO}}(\bar{u}, u^+) = f^+(\bar{u}) + f^-(u^+)$

$$\text{where } f^+(u) = F(0) + \int_0^u \max\{f'(v), 0\} dv, \quad f^-(u) = \int_0^u \min\{f'(v), 0\} dv$$

FV scheme: $\frac{d\bar{U}_i}{dt} = \frac{1}{\Delta x} [f(u_{i-\frac{1}{2}}^\pm, u_{i+\frac{1}{2}}^\pm) - f(u_{i+\frac{1}{2}}^\pm, u_{i-\frac{1}{2}}^\pm)]$, $u_{i\pm\frac{1}{2}}^\pm$ = WENO reconstructions at faces from mean values \bar{U}_i

FD scheme: $\frac{dU_i}{dt} = \frac{1}{\Delta x} [f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}}]$, $f_{i\pm\frac{1}{2}} = f(u_{i\pm\frac{1}{2}})$ = WENO reconstruction at faces from node values $F(U_i)$

$$[f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}] = f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}} + [f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}}] = f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}$$

WENO interpolation (3rd order case)

To find $U_{i+\frac{1}{2}}$ = face value, we interpolate the known values $\{U_{i-2}, U_{i-1}, U_i\}$ in $S_1 = \{x_{i-2}, x_{i-1}, x_i\}$ by a poly. $p_1(x) : p_1(x_j) = U_j$, then $U_{i+\frac{1}{2}} \approx p_1(x_{i+\frac{1}{2}}) \approx u(x_{i+\frac{1}{2}})$ with error $O(\Delta x^3)$ if u smooth.

$$\Rightarrow U_{i+\frac{1}{2}}^{(1)} = \frac{3}{8} U_{i-2} - \frac{5}{4} U_{i-1} + \frac{15}{8} U_i$$

Similarly, on stencil $\{U_{i-1}, U_i, U_{i+1}\}$, interpolation poly. $p_2(x)$, in stencil $S_2 = \{x_{i-1}, x_i, x_{i+1}\}$

$$U_{i+\frac{1}{2}}^{(2)} = -\frac{1}{8} U_{i-1} + \frac{3}{4} U_i + \frac{3}{8} U_{i+1}, \text{ also } O(\Delta x^3)$$

and on stencil $\{U_i, U_{i+1}, U_{i+2}\}$ with $p_3(x)$, on stencil $S_3 = \{x_i, x_{i+1}, x_{i+2}\}$

$$U_{i+\frac{1}{2}}^{(3)} = \frac{3}{8} U_i + \frac{3}{4} U_{i+1} + \frac{1}{8} U_{i+2}, \text{ also } O(\Delta x^3) \text{ if } u \text{ smooth on this stencil.}$$

On the overall stencil $\{U_{i-2}, U_{i-1}, U_i, U_{i+1}, U_{i+2}\}$, $p(x)$ of deg 4, gives

$$U_{i+\frac{1}{2}} = \frac{3}{128} U_{i-2} - \frac{5}{32} U_{i-1} + \frac{45}{64} U_i + \frac{15}{32} U_{i+1} - \frac{5}{128} U_{i+2}, O(\Delta x^5) \text{ if smooth on it.}$$

Important observation: $U_{i+\frac{1}{2}}$ can be written as a convex combination of the $U_{i+\frac{1}{2}}^{(1)}, U_{i+\frac{1}{2}}^{(2)}, U_{i+\frac{1}{2}}^{(3)}$:

$$U_{i+\frac{1}{2}} = \gamma_1 U_{i+\frac{1}{2}}^{(1)} + \gamma_2 U_{i+\frac{1}{2}}^{(2)} + \gamma_3 U_{i+\frac{1}{2}}^{(3)}, \gamma_1 + \gamma_2 + \gamma_3 = 1$$

$$\text{with } \gamma_1 = \frac{1}{16}, \gamma_2 = \frac{5}{8}, \gamma_3 = \frac{5}{16}$$

If $u(x)$ has a discontinuity point in $[x_{i-2}, x_{i+2}]$ then at least one of the 3rd order $U_{i+\frac{1}{2}}^{(k)}$ is still a 3rd order approximation, and $p(x)$ is monotone in the interval containing the discontinuity!

ENO chooses one of the small stencils based on smoothness indicator measured by finite differences,

WENO chooses a convex combination of $U_{i+\frac{1}{2}}^{(1)}, U_{i+\frac{1}{2}}^{(2)}, U_{i+\frac{1}{2}}^{(3)}$: $U_{i+\frac{1}{2}} = w_1 U_{i+\frac{1}{2}}^{(1)} + w_2 U_{i+\frac{1}{2}}^{(2)} + w_3 U_{i+\frac{1}{2}}^{(3)}$

with $w_1 + w_2 + w_3 = 1, w_j \geq 0$, which is a monotone approximation,

and want: $w_j \approx \gamma_j$ if $u(x)$ smooth in the big stencil, $w_j \approx 0$ if $u(x)$ has jump in stencil j but is smooth in the other small stencils.

Can be shown that as long as $w_j = \gamma_j + O(\Delta x^2)$ the WENO approximation $U_{i+\frac{1}{2}}$ is 5th order accurate if $u(x)$ is smooth in big stencil, and it is non-oscillatory, at least 3rd order if $u(x)$ is discontinuous in big stencil

Choice of weights w_j : $w_j = \frac{\bar{w}_j}{\bar{w}_1 + \bar{w}_2 + \bar{w}_3}$, where $\bar{w}_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^2}$, $\beta_j = \sum_{l=1}^{k=\deg p_j(x)=2} \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^l}{dx^l} p_j(x) \right)^2 dx, j=1, 2, 3$
 $\varepsilon = 10^{-6} > 0$ smoothness indicators

$$\beta_1 \text{ can be written out: } \beta_1 = \frac{1}{3} [4 U_{i-2}^2 - 19 U_{i-2} U_{i-1} + 25 U_{i-1}^2 + 11 U_{i-2} U_i - 31 U_{i-1} U_i + 10 U_i^2]$$

$$\beta_2 = \dots$$

$$\beta_3 = \dots$$

WENO reconstruction: via (piecewise) polynomials that preserve averages!

knowing cell averages \bar{U}_j find approximation for face values $u_{i+\frac{1}{2}}$

$$\text{Let } v(x) = \text{primitive of } u(x) = \int_{x-\frac{1}{2}}^x u(s) ds \quad \text{Then } v(x_{i+\frac{1}{2}}) = \sum_{l=0}^i \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} u(s) ds = \sum_{l=0}^i \Delta x \cdot \bar{U}_l$$

$\leftarrow \text{any fixed value}$

so knowing cell averages \bar{U}_j we also know face values of the primitive, $v(x_{i+\frac{1}{2}})$

Construct interpolation poly's for $v(s)$ and approximate $u(x) = v'(s)$ by $P'(x)$:

On stencil $S_1 = \{I_{i-2}, I_{i-1}, I_i\}$: $P_1(x)$ = interpolant of deg ≤ 3 of v at $x_{j+\frac{1}{2}}$, $j = i-3, i-2, i-1, i$

$$P_1(x) = P'_1(x) = \text{poly of deg } \leq 2 \text{ reconstructs } u(x): (P_1)_j = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} P_1(s) ds = \bar{U}_j,$$

$$\text{take } u_{i+\frac{1}{2}}^{(1)} = P_1(x_{i+\frac{1}{2}}) \quad j = i-2, i-1, i$$

$$\Rightarrow u_{i+\frac{1}{2}}^{(1)} = \frac{1}{3} \bar{U}_{i-2} - \frac{7}{6} \bar{U}_{i-1} + \frac{11}{6} \bar{U}_i \approx u(x_{i+\frac{1}{2}}) + O(\Delta x^3) \text{ if } u \text{ smooth in } S_1$$

Similarly on $S_2 = \{I_{i-1}, I_i, I_{i+1}\}$: P_2 interpolates v , $P'_2(x) = P_2'(x)$ reconstructs u on S_2

$$\Rightarrow u_{i+\frac{1}{2}}^{(2)} = -\frac{1}{6} \bar{U}_{i-1} + \frac{5}{6} \bar{U}_i + \frac{1}{3} \bar{U}_{i+1} \approx u(x_{i+\frac{1}{2}}) + O(\Delta x^3) \text{ if } u \text{ smooth in } S_2$$

and on $S_3 = \{I_i, I_{i+1}, I_{i+2}\}$: P_3 for v , $P'_3(x) = P_3'(x)$ for u on S_3

$$\Rightarrow u_{i+\frac{1}{2}}^{(3)} = \frac{1}{3} \bar{U}_i + \frac{5}{6} \bar{U}_{i+1} - \frac{1}{6} \bar{U}_{i+2} \approx u(x_{i+\frac{1}{2}}) + O(\Delta x^3) \text{ if } u \text{ smooth in } S_3$$

On big stencil $S = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\} = S_1 \cup S_2 \cup S_3$: P for v , $P'(x) = P'_1 v'$ for u

$$\Rightarrow u_{i+\frac{1}{2}} = \frac{1}{30} \bar{U}_{i-2} - \frac{13}{60} \bar{U}_{i-1} + \frac{47}{60} \bar{U}_i + \frac{9}{20} \bar{U}_{i+1} - \frac{1}{20} \bar{U}_{i+2} \approx u(x_{i+\frac{1}{2}}) + O(\Delta x^5)$$

$$= \sum_{j=1}^3 \gamma_j u_{i+\frac{1}{2}}^{(j)} \quad \text{with } \gamma_1 = \frac{1}{10}, \gamma_2 = \frac{6}{10}, \gamma_3 = \frac{3}{10} \quad \text{convex combination}$$

WENO choice for $u_{i+\frac{1}{2}} := \sum_{j=1}^3 w_j u_{i+\frac{1}{2}}^{(j)}$ with $w_j \geq 0$, $\sum w_j = 1$, $w_j \approx 0$ if u has jump

$$\Rightarrow w_j = \frac{\bar{w}_j}{\bar{w}_1 + \bar{w}_2 + \bar{w}_3} \quad \text{where } \bar{w}_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^2}, \quad \beta_j = \sum_{l=1}^k \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[\frac{d}{dx^l} P_j^{(x)} \right]^2 dx, \quad j = 1, 2, 3$$

$$\text{Explicit formula for } \beta_j: \quad \beta_1 = \frac{13}{12} (\bar{U}_{i-2} - 2\bar{U}_{i-1} + \bar{U}_i)^2 + \frac{1}{4} (\bar{U}_{i-2} - 4\bar{U}_{i-1} + 3\bar{U}_i)^2$$

$$\beta_2 = \frac{13}{12} (\bar{U}_{i-1} - 2\bar{U}_i + \bar{U}_{i+1})^2 + \frac{1}{4} (\bar{U}_{i-1} - \bar{U}_{i+1})^2$$

$$\beta_3 = \frac{13}{12} (\bar{U}_i - 2\bar{U}_{i+1} + \bar{U}_{i+2})^2 + \frac{1}{4} (3\bar{U}_i - 4\bar{U}_{i+1} + \bar{U}_{i+2})^2$$