

## High resolution Shock Capturing Schemes

for hyperbolic conservation laws:  $\frac{\partial \vec{u}}{\partial t} + \nabla \cdot \vec{F}(\vec{u}) = 0$  (system:  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$  in  $\mathbb{R}^n$ )

are driven by the need to solve the Euler equations (inviscid Navier-Stokes), the Navier-Stokes, and turbulence N-S, and a myriad variations...

These problems are nonlinear and shocks may develop ("surfaces" of "discontinuities" i.e. very abrupt changes in some components of  $\vec{u}$ ). They are by far the toughest (and largest) computational problems. They are "moving boundary problems", like phase transitions!

The mathematical theory was developed mainly in the 60's - 80's

and numerics in the 70's - 90's, to the point that the Boeing 777 was designed "on the computer" (early 90's), the first plane <sup>so designed</sup> and a triumph of computational science.

Model problem: 1-D scalar:  $u_t + F(u)_x = 0$

Shock capturing means we want to find the location of a shock: from  $\vec{u}$  without "tracking" it (shock location does not enter the formulation explicitly, the calculation does not rely on an explicit representation of shock location), (jump condition is not used to solve for  $\Sigma(t)$ )

Essential requirements:

1. Monotonicity preserving; if  $U^n$  is monotone mesh function, so is  $U^{n+1}$  hence
2. Non-oscillatory: no spurious oscillations
- and 3. Local extremum diminishing: local maxima do not increase and local minima do not decrease
4. Higher order

The solution of  $u_t + F(u)_x = 0$  has the property that the total variation

$$TV[u] = \int_{-\infty}^{\infty} \left| \frac{du}{dx} \right| dx$$

does not increase, provided any discontinuity in  $u$  satisfies a so-called entropy condition, which in fact ensures  $u$  will be a physical solution. This PDE admits many "weak" solutions, but only one satisfying an entropy condition (an inequality, which in the gas dynamics case ensures entropy <sup>decrease</sup> does not increase).

The space  $BV$  of function of Bounded total Variation ( $TV[u] < \infty$ ) is the natural space for solutions of hyperbolic conserv. laws, allowing only jump discontinuities and finite total length. The theory of such weak solutions was developed in the 60's - 70's (Dafermos, DiPerna, ...)

## High resolution schemes for $u_t + F(u)_x = 0$

core developments

1959

Godunov: FV Riemann solvers

1960s-70s

Flux limiters, entropy sol, positivity, monotonicity  
van Leer, Roe, ...

1983

Harten: TVD schemes  $TV(U^{n+1}) \leq TV(U^n)$

$\Rightarrow$  monotonicity preserving  $\Rightarrow$  no spurious oscillations

But can be only 1st order, ... lead to TVB

Flux limiters produce TVD schemes

1987

Harten-Osher-Enquist: ENO :  $TV(U^{n+1}) \leq TV(U^n) + \mathcal{O}(\Delta x^p)$

Sethian-Osher: Level Set Methods for capturing moving fronts

soon after

Shu-Osher: WENO (Weakly ENO)

Reconstruct - Update - Average paradigm (grew from Godunov approach)

$$\hat{F}(u) \approx h(\bar{u}_{i-\frac{1}{2}}, \bar{u}_{i+\frac{1}{2}}) \quad \text{monotone flux function}$$

$\bar{u}_{i \pm \frac{1}{2}}$ : reconstructed face values from high-order polynomials that

1. from cell averages  $U_i^n$ , reconstruct smooth approximate sol  $\tilde{u}(x, t_n)$  (high order interpolant) preserve averages

2. advance  $\tilde{u}$  exactly or via approximate Riemann solvers

to estimate fluxes at cell faces:  $F(\tilde{u}_{i \pm \frac{1}{2}})$

3. find average of updated  $\tilde{u}$  over each cell to get new  $U_i^{n+1}$

mid 2000

Gottlieb-Shu: SSP time steppers

2010

Zhang-Shu: Max-Principle-Satisfying and monotonicity-preserving high order  
for FV, FD, LDG with SSP-RK time stepping

# Flux Limiter methods for advection

1-D (van Leer's method) TVD schemes

LeVeque, SIAM J. Num. Anal., 1996

Want to "combine" upwind and Lax-Wendroff hoping to create a better overall method.  
for the linear advection  $u_t + (vu)_x = 0$

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} [ F_{i-\frac{1}{2}}^n - F_{i+\frac{1}{2}}^n ]$$

Upwind flux:  $F_{i-\frac{1}{2}}^{\text{up}} = \begin{cases} vU_{i-1} & \text{if } v > 0 \\ vU_i & \text{if } v < 0 \end{cases}$

Lax-Wendroff flux:  $F_{i-\frac{1}{2}}^{\text{LW}} = \frac{1}{2}v[U_{i-1} + U_i] - \frac{\Delta t}{2\Delta x}v^2[U_i - U_{i-1}]$

(check:  $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} \left[ \frac{1}{2}v(U_{i-1} + U_i) - \frac{\Delta t^2}{2\Delta x^2}v^2(U_{i-1} - U_i) - \frac{\Delta t^2}{2\Delta x^2}v^2(U_{i+1} - U_i) \right]$   
 $= U_i^n + \frac{v\Delta t}{2\Delta x} [U_{i-1} - U_{i+1}] + \left( \frac{v\Delta t}{\Delta x} \right)^2 \frac{1}{2} [U_{i-1} - 2U_i + U_{i+1}]$ )

$F^{\text{LW}}$  can be written as  $F^{\text{up}} + \text{correction}$ ;

$$F_{i-\frac{1}{2}}^{\text{LW}} = F_{i-\frac{1}{2}}^{\text{up}} + \frac{1}{2}|v| \left( 1 - \left| \frac{v\Delta t}{\Delta x} \right| \right) [U_i - U_{i-1}]$$

Flux-limiter method:  $F_{i-\frac{1}{2}} = F_{i-\frac{1}{2}}^{\text{up}} + \frac{|v|}{2} \left( 1 - \left| \frac{v\Delta t}{\Delta x} \right| \right) [U_i - U_{i-1}] \cdot \Phi_{i-\frac{1}{2}}$

where

$$\Phi_{i-\frac{1}{2}} = \text{"limiter"} = \varphi(\theta_{i-\frac{1}{2}}), \quad \theta_{i-\frac{1}{2}} = \frac{U_i - U_{i-1}}{U_i - U_{i-1}}, \quad I = \begin{cases} i-1, & \text{if } v > 0 \\ i+1, & \text{if } v < 0 \end{cases}$$

So the limiter depends on the local nature of the solution  $\rightarrow \underset{i-1}{\times} + \underset{i}{\times} + \underset{i+1}{\times} + \underset{i+2}{\times}$

since  $\theta_{i-\frac{1}{2}}$  is the ratio of slopes at the upwind face to the current face,

Note that if  $\Phi=0$  we get Upwind, if  $\Phi=1$  we get Lax-Wendroff.

class 13

(INT) (bottom right)  $\theta = \frac{U_{i+1} - U_i}{\Delta x}$  (bottom left)  $\theta = \frac{U_{i-1} - U_i}{\Delta x}$

Some popular limiters:

$$\text{Mlimod: } \varphi(\theta) = \max\{0, \min(1, \theta)\}$$

$$\text{superbee: } \varphi(\theta) = \max\{0, \min(1, 2\theta), \min(2, \theta)\}$$

$$\text{van Leer: } \varphi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}$$

$$\text{monotonized centered(van Leer): } \varphi(\theta) = \max\{0, \min\left(\frac{1+\theta}{2}, 2, 2\theta\right)\}$$

The last one seems to work well in general. It produces the "centered approximation"

$$(U_i - U_{i-1})\varphi(\theta_{i+1/2}) = \frac{1}{2} [(U_i - U_{i-1}) + (U_I - U_{I-1})]$$

unless this is larger than  $2(U_i - U_{i-1})$  or  $2(U_I - U_{I-1})$ , in which case it is appropriately limited, and  $\varphi=0$  if  $\theta < 0$ .

Limiters are designed to preserve monotonicity and not increase the total variation of  $U$  at any time-step. These are the "TVD" schemes (total variation diminishing), developed in mid-1970's to mid-1980's.

Limiters are used to suppress possible oscillations near discontinuities, and NOT needed where solution is smooth.

# Slope-limiter Methods for 1-D advection

LeVeque, SIAM J. Num. Anal., 1996

A better, more geometric way to view flux-limiter methods is as follows, which points the way to higher order schemes.

General procedure: reconstruct-solve-average (Godunov style)

- From the given cell averages  $\{U_i^n\}$  "reconstruct" a function  $\tilde{u}(x, t_n)$ .
- Solve the advection equation exactly with these data over  $\Delta t$  to get

$$\tilde{u}(x, t_{n+1}) = \tilde{u}(x - v\Delta t, t_n) \quad [\text{for } u_t + (vu)_x = 0]$$

- or by exact Riemann solver or by approximate Riemann solver
- Average this updated  $\tilde{u}$  over each cell to get

$$U_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}(x, t_{n+1}) dx.$$

Different choices for  $\tilde{u}$  give different methods; For example:

- I. Piecewise constant  $\tilde{u}$ ; Take  $\tilde{u}(x, t_n) = U_i^n$ ,  $x_{i-1/2} \leq x < x_{i+1/2}$ .

Then we find

$$U_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}(x, t_{n+1}) dx = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}(x - v\Delta t, t_n) dx = \frac{1}{\Delta x} \int_{\xi=x_{i-1/2}-v\Delta t}^{\xi=x_{i+1/2}-v\Delta t} \tilde{u}(\xi, t_n) d\xi =$$

$$= \frac{1}{\Delta x} \int_{x_{i-1/2}-v\Delta t}^{x_{i-1/2}} U_{i-1}^n d\xi + \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}-v\Delta t} U_i^n d\xi \quad (\text{assuming } v > 0, v\Delta t \leq \Delta x)$$

$$= \frac{1}{\Delta x} U_{i-1}^n \cdot v\Delta t + \frac{1}{\Delta x} U_i^n \cdot [x_{i+1/2} - x_{i-1/2} - v\Delta t] = \frac{v\Delta t}{\Delta x} U_{i-1}^n + U_i^n - \frac{v\Delta t}{\Delta x} U_i^n$$

$$= U_i^n + \frac{v\Delta t}{\Delta x} [U_{i-1}^n - U_i^n] \quad \text{which is exactly the Upwind scheme!}$$

II. Piecewise linear  $\tilde{u}$  with mean value  $U_i^n$  and some slope  $\sigma_i$  in cell  $i$ :

$$\tilde{u}(x, t_n) = U_i^n + \sigma_i (x - x_i), \quad x_{i-1/2} \leq x \leq x_{i+1/2}$$

Then we get: assuming  $v > 0, v\Delta t \leq \Delta x$ :

$$\begin{aligned} U_i^{n+1} &= \frac{1}{\Delta x} \int_{x_{i-1/2}-v\Delta t}^{x_{i-1/2}} [U_{i-1}^n + \sigma_{i-1} (\tilde{y} - x_{i-1})] d\tilde{y} + \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}-v\Delta t} [U_i^n + \sigma_i (\tilde{y} - x_i)] d\tilde{y} = \\ &= \frac{v\Delta t}{\Delta x} U_{i-1}^n + \frac{\sigma_{i-1}}{\Delta x} \int_{x_{i-1/2}-v\Delta t}^{x_{i-1/2}-x_{i-1}} y dy + U_i^n - \frac{v\Delta t}{\Delta x} U_i^n + \frac{\sigma_i}{\Delta x} \int_{x_{i-1/2}-x_i}^{x_{i+1/2}-x_i-v\Delta t} y dy \\ &= U_i^n + \frac{v\Delta t}{\Delta x} [U_{i-1}^n - U_i^n] + \frac{\sigma_{i-1}}{2\Delta x} \left[ \left( \frac{\Delta x}{2} \right)^2 - \left( \frac{\Delta x}{2} - v\Delta t \right)^2 \right] + \frac{\sigma_i}{2\Delta x} \left[ \left( \frac{\Delta x}{2} - v\Delta t \right)^2 - \left( \frac{\Delta x}{2} \right)^2 \right] \\ &= \text{upwind} \quad + \frac{\sigma_{i-1}}{2\Delta x} \left[ + \Delta x v\Delta t - (v\Delta t)^2 \right] + \frac{\sigma_i}{2\Delta x} \left[ - \Delta x v\Delta t + (v\Delta t)^2 \right] \\ &= " \quad + \frac{v\Delta t}{2} \left( 1 - \frac{v\Delta t}{\Delta x} \right) \sigma_{i-1} + \frac{v\Delta t}{2} \left( \frac{v\Delta t}{\Delta x} - 1 \right) \sigma_i \\ \Rightarrow U_i^{n+1} &= U_i^n + \frac{v\Delta t}{\Delta x} [U_{i-1}^n - U_i^n] + \frac{1}{2} v\Delta t \left( 1 - \frac{v\Delta t}{\Delta x} \right) [\sigma_{i-1} - \sigma_i] \end{aligned}$$

We can write this in terms of fluxes as  $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} [F_{i-1/2} - F_{i+1/2}]$  with

$$F_{i-1/2} = F_{i-1/2}^{\text{up}} + \frac{1}{2} |v| \Delta x \left( 1 - \frac{|v| \Delta t}{\Delta x} \right) \sigma_I, \quad I = \begin{cases} i-1, & \text{if } v > 0 \\ i, & \text{if } v < 0 \end{cases}$$

In particular, choosing  $\sigma_I = \frac{U_i - U_{i-1}}{\Delta x}$ , we obtain

$$F_{i-1/2} = F_{i-1/2}^{\text{up}} + \frac{1}{2} |v| \left( 1 - \frac{|v| \Delta t}{\Delta x} \right) [U_i - U_{i-1}] = F_{i-1/2}^{\text{LW}} = \text{Lax-Wendroff!}$$

All the flux-limiter methods can be interpreted as slope-limiter methods with slope:

$$\sigma_I = \frac{U_i - U_{i-1}}{\Delta x} \Phi_i$$

Higher order reconstructions  $\tilde{u}$  lead to higher order schemes!

such as ENO and WENO reconstructions (3<sup>rd</sup> or 5<sup>th</sup> order)

nonoscillatory, minimal num. diffusion.

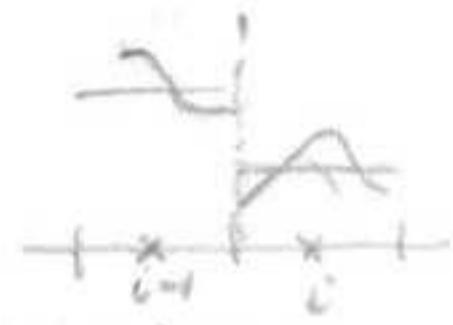
## Conservative FV schemes for $u_t + F(u)_x = 0$

Godunov type schemes

$$\text{FV conservative scheme: } \bar{U}_i^{n+1} = \bar{U}_i^n + \frac{\Delta t}{\Delta x} [\hat{F}(\tilde{u}_{i-\frac{1}{2}}) - \hat{F}(\tilde{u}_{i+\frac{1}{2}})]$$

(exact)

$\bar{U}_i$  = mean value of  $u$ , but face values  $u_{i \pm \frac{1}{2}}$  not known! and may jump



we need to approximate the fluxes by a numerical flux function  $h(v, w)$

$$\hat{F}_{i \pm \frac{1}{2}} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} F(u(x_{i \pm \frac{1}{2}}, t)) dt \approx h(u_{i \pm \frac{1}{2}}^-, u_{i \pm \frac{1}{2}}^+) \quad \text{with } u_{i \pm \frac{1}{2}}^\pm \text{ high order approximations of } u(x_{i \pm \frac{1}{2}}, t_n)$$

face values

Desired properties of num. flux  $h(v, w)$ :

1. Lipschitz cont's in each variable
2. consistent:  $h(u, u) = F(u)$
3. monotone:  $h(\uparrow, \downarrow)$ : nondecreasing in 1<sup>st</sup> argument and nonincreasing in 2<sup>nd</sup> argument

Simplest such is the Lax-Friedrichs flux:  $h(v, w) = \frac{1}{2} [F(v) + F(w) - a(w-v)]$ ,  $a = \max_u |F'(u)|$

$$\left[ \text{check: } h(u, u) = \frac{1}{2} [F(u) + F(u) - 0] = F(u) \right. \\ \left. \frac{\partial h}{\partial v} = \frac{1}{2} [F'(v) + a] \geq 0 \because \uparrow \text{ in } v, \quad \frac{\partial h}{\partial w} = \frac{1}{2} [F'(w) - a] \leq 0 \because \downarrow \text{ in } w \right]$$

A more complicated is the HLL-E flux, and others

High order approximations of  $u(x_{i \pm \frac{1}{2}}, t_n)$ :

From the mean values  $\bar{U}_i^n$ , "reconstruct"  $u(x, t_n)$  in  $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$  (approximate) (e.g. a k-th degree polynomial  $P_i(x)$  in  $I_i$ ) preserving the average  $\bar{U}_i^n : \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{u}(x) dx = \bar{U}_i^n, i=1, \dots, M$