

Max Principle: $\min_x u(x, 0) \leq u(x, t) \leq \max_x u(x, 0) \quad \forall x, t$

obeyed by sol of parabolic PDEs and by the (unique) entropy weak sol of scalar $u_t + \nabla \cdot F(u) = 0$
 but not by systems of conservation laws

- ⇒ positivity
- ⇒ monotonicity
- ⇒ uniqueness

Want: high order monotone schemes (∴ non-oscillatory)

but
Godunov (1959) order barrier: A ^{positive} monotone linear scheme is at most 1st order accurate!

The 1st order upwind scheme is "optimal" (most accurate)!

Only hyper is non-linear high order

Order barrier for diffusion schemes to be positive; cannot be higher than 2nd order

and the centered scheme is optimal

so, the ^{FV} schemes we studied are the optimal monotone schemes!

monotone \Rightarrow TVD \Rightarrow monotonicity preserving

(scheme for hyperbolic conservation laws, LeVeque, Conserv. Laws, 1990)

Monotone property:

of conserv. laws: $u(x,0) \leq v(x,0) \Rightarrow u(x,t) \leq v(x,t) \forall t > 0 \forall x$

Monotone scheme: $U_i^n \leq V_i^n \forall i \Rightarrow U_i^{n+1} \leq V_i^{n+1} \forall i$ for any grid functions U_i, V_i

If scheme can be written as $U_i^{n+1} = \mathcal{J}\mathcal{C}(U_{i-k}^n, \dots, U_i^n, \dots, U_{i+l}^n)$

it is monotone (positive) iff $\mathcal{J}\mathcal{C}(\cdot)$ is an increasing function of all its arguments

So for $\mathcal{J}\mathcal{C}$ smooth: $\frac{\partial \mathcal{J}\mathcal{C}}{\partial U_j} \geq 0 \forall j = -k, \dots, 0, \dots, l$

TVD scheme: $TV(U^{n+1}) \leq TV(U^n)$ for any grid function U is, it does not increase TV
i.e. no new oscillations can develop (non-oscillatory).

where $TV(U) = \sum_{i=1}^M |U_i - U_{i-1}|$ = total variation of grid function U

Total variation of a function $f(x)$ on $[a,b]$: $TV(f) = \sup \sum_{i=1}^M |f(x_i) - f(x_{i-1})|$ over all partitions of $[a,b]$
 $a = x_0 < \dots < x_M = b$

A monotone scheme converges to the unique entropy ^{condition-satisfying} solution of conservation law,
but a TVD scheme may converge to some other weak solution (to a wrong solution).

Monotonicity Preserving scheme: U_i^n monotone function of $i \Rightarrow U_i^{n+1}$ is also monotone function of i

nice properties: 1. no new local extrema can appear at later times

2. value of solution at a local min is non-decreasing

and " " " " max " non-increasing

Harten (1983): monotone \Rightarrow TVD \Rightarrow monotonicity preserving

but

Godunov (1959) order barrier: A ^{positive} monotone linear ^{advection} scheme is at most first order accurate!

And the 1st order upwind scheme is "optimal" (most accurate)!

SSP = Strong Stability Preserving schemes

means $\|U^{n+1}\| \leq \|U^n\| \quad \forall n$ in some norm

Necessary for hyperbolic, especially for nonlinear with shocks

Developed by Gottlieb-Ketcheson-Shu 2009, book 2011

Many RK schemes are not SSP, need $a_{ij} \geq 0, b_k \geq 0$, restriction on Δt :
classical RK4 is not SSP!
 $\Delta t \leq c \cdot \Delta t_{\text{expl}}$
 $c \leq s - p + 1$

explicit optimal SSP RK $(2, 2)$: $U^{(1)} = U^n + \tau \cdot F(U^n), U^{n+1} = \frac{1}{2} U^n + \frac{1}{2} U^{(1)} + \frac{\tau}{2} F(U^{(1)})$

SSP RK $(3, 3)$: $U^{(1)} = U^n + \tau \cdot F(U^n), U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{\tau}{4} F(U^{(1)}),$
 $U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{(2)} + \frac{2}{3} \tau F(U^{(2)})$

SSP RK $(4, 4)$: none is SSP

$(5, 4)$: ... messy coefficients...

implicit SSP RK must have order $p \leq 6$, and they have $c \leq 2$, so not competitive!