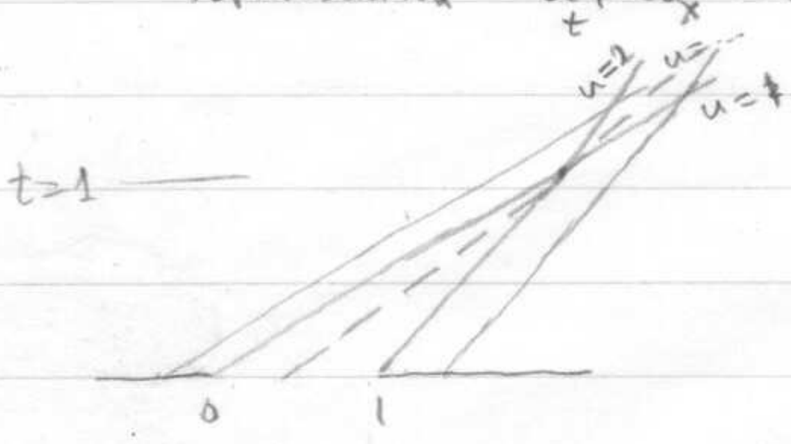
Jan 24

Vanishing viscosity solution
~~Weak solution formulation~~ of Burgers' equation

What happens after $t=1$ in our example? Does the world come to an end?

Well, only the world of classical solutions! The standard (pointwise) PDE formulation $u_t + uu_x = 0$ no longer makes sense since u_x blows up!



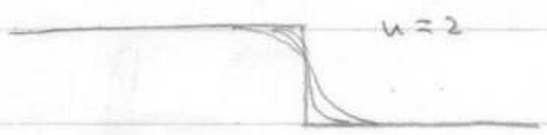
Continuing the characteristics beyond $t=1$, we see that the "solution" tries to become multivalued

In some physical situations this may make sense, for a while anyway, until the wave "breaks". Example: waves on a sloping beach.

In most situations, multivalued solutions do <u>not</u> make sense, you can't have multivalued density of a gas!

Remember that the Euler equations are a "simplified" model of the Navier-Stokes equations when viscosity is negligible. So $u_t + uu_x = 0$ is the limiting case of the viscous $u_t + uu_x = \nu u_{xx}$ as the viscosity $\nu \to 0$.

For very small ν, the diffusion term νu_{xx} is negligible <u>as long as u is smooth</u>! and we see advection. But as the profile becomes very steep, u_x and u_{xx} becomes very large, so diffusion kicks in, keeping the solution smooth and preventing the blow up. The smaller the viscosity ν, the steeper the solution can become, and in the limit $\nu \to 0$ we get a discontinuous solution



The viscous solution continues to be smooth and steep and simply propagates, so we can expect that the inviscid solution becomes ⌐ and propagates. This is the behavior we hope to capture by solving the inviscid eq hyperbolic PDE, called the "vanishing viscosity solution".

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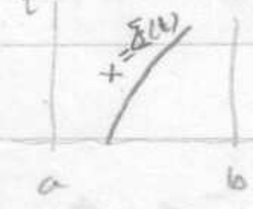
We have enlarged the concept of solution to admit non-smooth u.
The danger is that if we overdo it, we may loose uniqueness.
In fact, this is the case here. Our weak solution concept may allow several weak solutions of the problem, so we need to impose an "entropy inequality" to pick out a unique solution! (guided by physics)

Solution after the shock forms

Our weak sol. makes sense even after a shock forms, so let's try to find it.
From the "vanishing viscosity" approach, we know what to expect: a shock forms, a single curve $x = \Sigma(t)$ across which u jumps from $u = 2 = u^-$ on its left to $u = 1 = u^+$ on its right. Can we find such a curve? assume a single sharp smooth shock curve

Consider a weak sol. u which is smooth in Q except possibly across a C^1-curve $x = \Sigma(t)$ where it has a jump discontinuity. For any $\varphi \in C_0^\infty(Q) \subset \Phi$ we have



$$0 = \iint_Q \{ u\varphi_t + F u_x \}\, dx\,dt = \iint_{Q^-} + \iint_{Q^+} \quad \text{and since } u \text{ is smooth in } Q^\pm$$

$$= \left(\int_{\partial Q^-} + \int_{\partial Q^+} \right) (u\varphi n_t + F\varphi n_x)\, ds - \underbrace{\left(\iint_{Q^-} + \iint_{Q^+} \right) (u_t + F_x)\varphi\, dx\,dt}_{=0 \text{ since } u \text{ is classical sol in } Q}$$

$\vec{n}\,ds = \langle n_x, n_t \rangle ds = \langle dt, -dx \rangle$

$$= \int_{x=\Sigma(t)} \left[u^- \varphi\,(-dx) + F(u^-)\varphi\, dt \right] - \int_{x=\Sigma(t)} \left[u^+ \varphi\,(-dx) + F(u^+)\varphi\, dt \right] + 0$$

($\varphi = 0$ on rest of $\partial Q^\pm$)

$$= \int_{x=\Sigma(t)} \varphi \cdot \left([u^+ - u^-]\, dx - [F(u^+) - F(u^-)]\, dt \right)$$

$$= \int_{x=\Sigma(t)} \varphi \cdot \left([\![u]\!]^+_- \frac{dx}{dt} - [\![F(u)]\!]^+_- \right) dt \qquad \forall\, \varphi \in C_0^\infty(Q)$$

and we obtain the

Rankine-Hugoniot or Shock condition or jump condition:

$$[\![u]\!]^+_- \Sigma'(t) = [\![F(u)]\!]^+_-$$

which in fact determines the speed of the shock. Note that this condition is built into the weak formulation, ie. a weak sol. will automatically satisfy it. Here we made it explicit under the assumption of a single sharp smooth shock curve.

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Remark: The jump condition holds for any smooth curve in Ω:

If u is cont's across the curve then $0 \cdot \Sigma' = [\![F]\!]^+_-$ $\Rightarrow$ F must be cont's too

If F is cont's across the curve then either u is also cont's or $\Sigma' = 0$ i.e. $\Sigma(t)$ is a vertical line at x (i.e. Σ does not move)

Construction of the shock solution in our example: After $t^* = 1$ we have:

$[\![u]\!]^+_- = u^+ - u^- = 1 - 2 = -1$

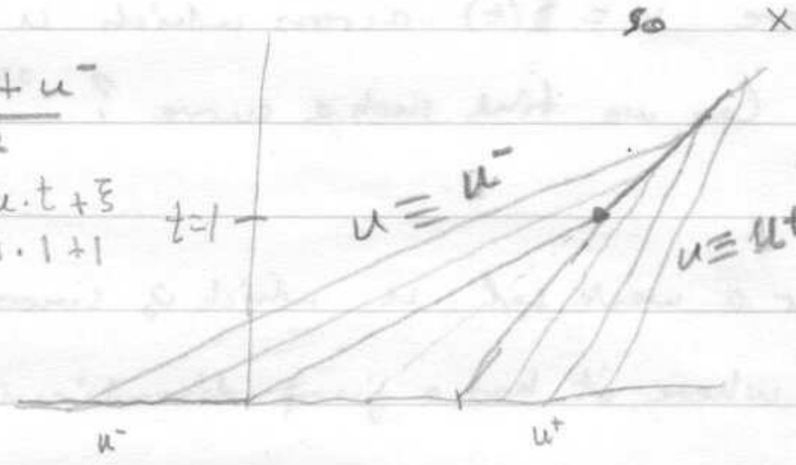
$[\![F(u)]\!]^+_- = \left[\!\left[\frac{u^2}{2} \right]\!\right]^+_- = \frac{1^2}{2} - \frac{2^2}{2} = \frac{1}{2} - 2 = -\frac{3}{2}$, so

$= \frac{(u^+)^2 - (u^-)^2}{2}$ $\quad (-1)\,\Sigma'(t) = -\frac{3}{2} \Rightarrow \Sigma'(t) = \frac{3}{2} \left(= \frac{2+1}{2}\right)$ thru $(x=2, t=1)$

in fact $= \frac{(u^+ - u^-)(u^+ + u^-)}{2}$

$\Rightarrow$ $\Sigma'(t) = \frac{u^+ + u^-}{2}$ so $x = \Sigma(t) = \frac{1+3t}{2}$

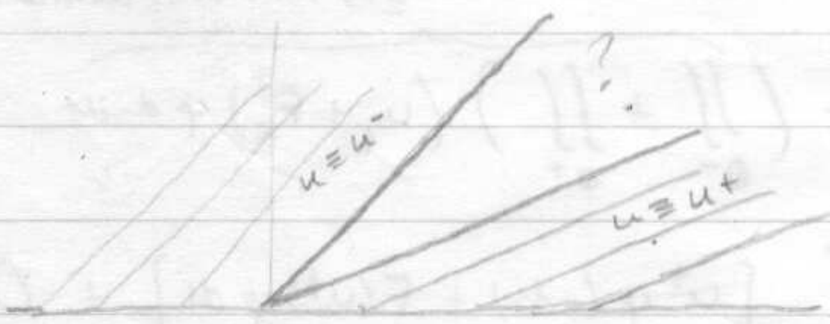
$\Sigma(1) = 2 = u \cdot t + \xi$
$1 \cdot 1 + 1$



is the unique weak sol. of the example problem, which exists for all t.

Example of Rarefaction wave: $u_t + u u_x = 0$ $\quad (= 1)$

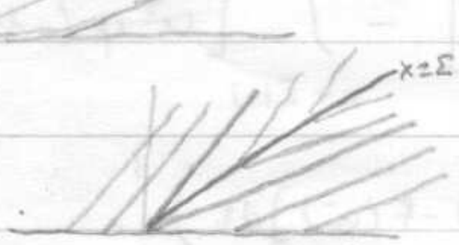
$u(x,0) = u_0(x) = \begin{cases} u^-, & x < 0 \\ u^+, & x > 0 \end{cases}$, $u^\pm$ = const. but $u^- < u^+$ $\quad 1 < 2$



Now the characteristics open up, don't intersect for $t > 0$!

In this case we can construct infinitely many weak sol's!

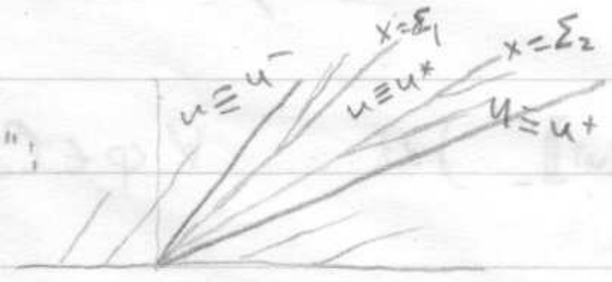
Solution with 1 "shock":



$\Sigma'(t) = \frac{u^+ + u^-}{2}$, $u = \begin{cases} u^-, & x < \Sigma(t) \\ u^+, & x > \Sigma(t) \end{cases}$

Solution with 2 "shocks":

pick any $u^- < u^* < u^+$



$\Sigma_1'(t) = \frac{u^- + u^*}{2}$

$\Sigma_2'(t) = \frac{u^* + u^+}{2}$

$u = \begin{cases} u^- \\ u^* \\ u^+ \end{cases}$

Solution with N "shocks": similarly

Solution without shocks; fan solution or rarefaction wave; $u = \begin{cases} u^-, & x < u^- t \\ \frac{x}{t}, & u^- t < x < u^+ t \\ u^+, & x > u^+ t \end{cases}$

Which one is physically correct? The one satisfying the "entropy inequality": $F'(u^+) \le \Sigma' \le F'(u^-)$

which for Burgers: $u^+ \le \Sigma' \le u^-$ so excludes shocks when $u^- < u^+$ and allows only the rarefaction wave!

Entropy solution of $u_t + F(u)_x = 0$

Def: A discontinuity propagating with speed $\Sigma'(t)$ given by the Rankine-Hugoniot condition satisfies the entropy condition if $\left([u]^+_- \Sigma'(t) = [F(u)]^+_- \right)$

$$F'(u^-) \geq \Sigma'(t) \geq F'(u^+)$$

Note that $u_t + F'(u) u_x = 0$, so $F'(u)$ is the advection speed at which the signal propagates. So the entropy condition says the propagation speed on the left should by higher than the shock speed and on the right lower, so that characteristics run into the shock from both sides

[valid only for convex $F(u)$]

Several characterizations of "entropy solutions" have been developed. A general one is:

Def. u is an entropy solution of $u_t + F(u)_x = 0$ if for any convex function $\eta(u)$, $\eta''(u) > 0$ and $\psi(u)$ such that $\psi'(u) = \eta'(u) F'(u)$ the entropy inequality

$$\eta(u)_t + \psi(u)_x \leq 0 \quad \text{holds}$$

in the weak sense:

$$\int_{-\infty}^{\infty} \int_0^{\infty} \{ \varphi_t \eta(u) + \varphi_x \psi(u) \} dt dx \leq - \int_{-\infty}^{\infty} \varphi \eta(u) \Big|_{t=0} dx \quad \forall \varphi \in C_0^\infty, \varphi \geq 0.$$

Kruskov Thm: (Kruzkov (1970)) The CP: $\begin{cases} u_t + F(u)_x = 0, & -\infty < x < \infty, \ t > 0 \\ u(x,0) = u_0(x), & \end{cases}$ $F \in C^1(\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R})$

has unique entropy (weak) solution $u \in L^\infty$ which satisfies:

1. For almost all $t > 0$, $\| u(\cdot, t) \|_{L^\infty} \leq \| u_0 \|_{L^\infty}$
2. If $u_0 \geq v_0$ a.e. in $\mathbb{R}$ then $u(x,t) \geq v(x,t)$ for a.a. x, a.a. $t > 0$ (monotonicity)
3. If $u_0 \in BV(\mathbb{R})$ then $u(\cdot, t) \in BV(\mathbb{R})$ for a.a. $t > 0$ and $TV[u(\cdot,t)] \leq TV[u_0]$ $\left(= \int_{-\infty}^{\infty} \left| \frac{\partial u_0}{\partial x} \right| dx \right)$ (TVD property)
4. If $u_0 \in L^1(\mathbb{R})$ then $\int_{\mathbb{R}} u(x,t) dx = \int_{\mathbb{R}} u_0(x) dx$ a.a. $t > 0$ (conservative)

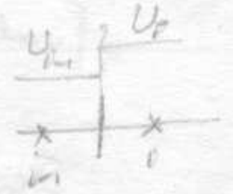
S.N. Kruzkov, Sbornik 10(2), 1970

Lax-Wendroff Theorem: If the numerical solution U of a consistent, conservative scheme converges to a function u in L^1 for a.a. $t>0$ as $\Delta x, \Delta t \to 0$, with $\frac{\Delta t}{\Delta x}$ fixed, then u is a weak solution of the Cauchy Problem.

Conservative schemes for $u_t + F(u)_x = 0$: $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left[\hat{F}^n_{i-1/2} - \hat{F}^n_{i+1/2}\right]$

Scheme in Conservation form : $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left[h(U_{i-1}, U_i) - h(U_i, U_{i+1})\right]$

for some "numerical flux function" $h(U_{i-1}, U_i) \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F\left(u(x_{i-1/2}, t)\right) dt$



which we want it to be Lipschitz continuous and monotone

1. consistency : $h(u,u) = F(u)$
2. monotonicity : $h(\uparrow, \downarrow)$

Simplest such num. flux is the Lax-Friedrichs flux : $h(u,v) = \frac{1}{2}\left[F(u) + F(v) - a(v-u)\right]$, $a = \max_u |F'(u)|$