

Burgers' Equation

Nonlinear scalar conservation laws: $u_t + (F(u))_x = 0$ $F(u)$
 has nonlinear flux $F(u)$ instead of $v \cdot u$, so now v depends on u .

Canonical example: $u_t + (\frac{1}{2} u^2)_x = 0$ inviscid Burgers' equation

[Burgers, 1948]

$u_t + (\frac{1}{2} u^2)_x = \nu u_{xx}$ viscous Burgers' equation

These are the model ^{problems for convection} nonlinear ^{advection} and ^{convection} advection-diffusion, $\nu = \text{viscosity}$.

The nonlinear flux $F(u) = \frac{1}{2} u^2$ is important because u^2 is exactly the nonlinearity in the momentum equation of the Euler equations of Gas Dynamics:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + (\rho v^2 + P)_x &= 0 \quad \text{or} \quad \vec{u}_t + (\vec{F}(\vec{u}))_x = 0, \quad \vec{u} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} \rho v \\ \rho v^2 + P \\ v(E+P) \end{bmatrix} \\ E_t + (v(E+P))_x &= 0 \end{aligned}$$

Thus, $u_t + u u_x = 0$ is the simplest physically relevant nonlinear ^{con} advection eqn. and $u_t + u u_x = \nu u_{xx}$ is the simplest ^{con} advection-diffusion eqn. modeling the nonlinearity in the Navier-Stokes equations.

Method of characteristics: $u_t + u u_x = 0 \implies \frac{dt}{1} = \frac{dx}{u} \implies \frac{dx(t)}{dt} = u(x(t), t)$

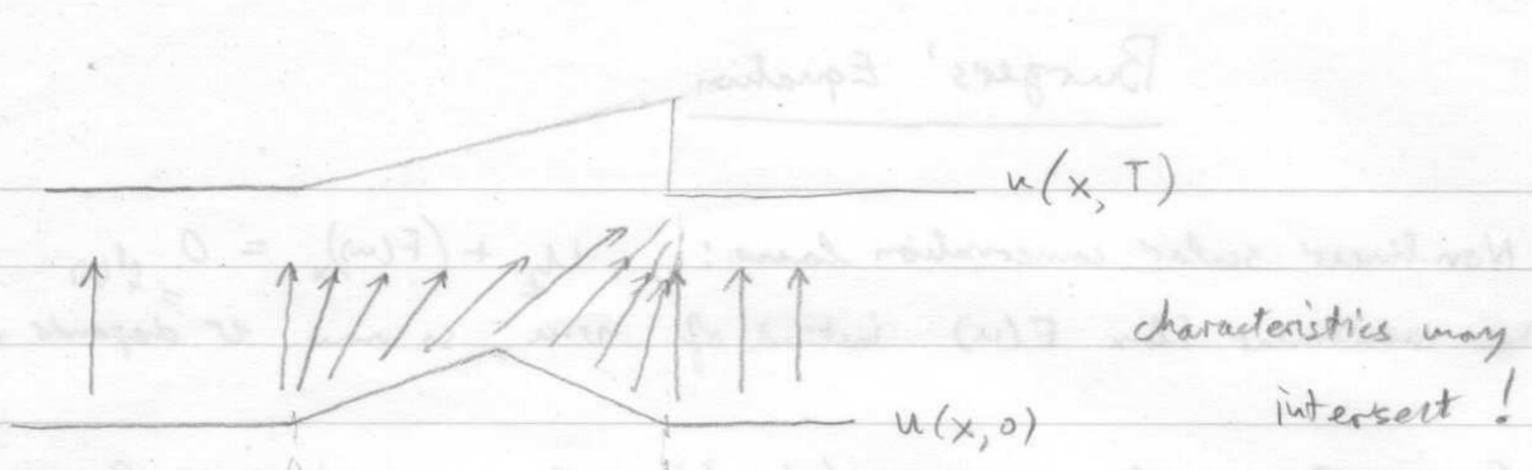
$$\implies \frac{d}{dt} u(x(t), t) = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} = u_x u + u_t = 0 \quad \text{PDE}$$

so u is constant ^(in time) along characteristics: $u(x(t), t) = u_0(\xi) \quad \forall t$
 so $\frac{dx}{dt} = u_0(\xi) \implies x(t) = u_0(\xi) \cdot t + \xi$ are straight lines on $x-t$ plane emanating from $x = \xi$ at $t = 0$,

but not parallel, since the slope $\frac{1}{u_0(\xi)}$ depends on the value of $u_0(\xi)$.
 (unless $u_0(x) \equiv \text{const}$)

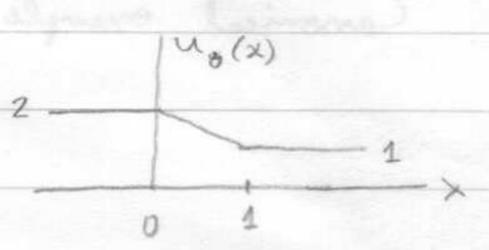
The solution of the Cauchy Problem:
$$\begin{cases} u_t + u u_x = 0 & -\infty < x < \infty \\ u(x, 0) = u_0(x) & \end{cases}$$

is given implicitly by $u(x, t) = u_0(x - ut)$, $= u_0(\xi), \xi = x - u_0(\xi)t$ assuming of course that $u_0(x)$ is smooth. Nevertheless, there may be trouble:



Example:
$$\begin{cases} u_t + uu_x = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

$$u_0(x) = \begin{cases} 2, & x \leq 0 \\ 2-x, & 0 < x < 1 \\ 1, & 1 \leq x \end{cases}$$

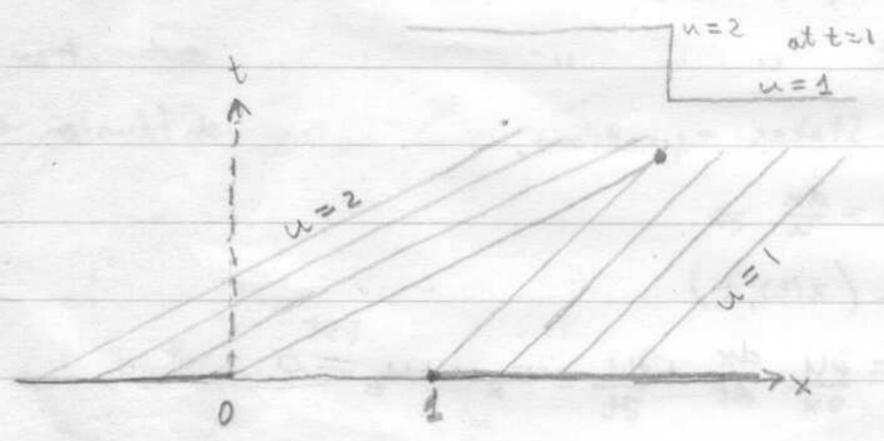


Solution: Characteristics: $\frac{dx}{dt} = u(x, t) \Rightarrow x = u_0(\xi)t + \xi$
 $u \equiv \text{const. on characteristics} \Rightarrow u(x, t) = u_0(\xi) = \begin{cases} 2, & \xi \leq 0 \\ 2-\xi, & 0 < \xi < 1 \\ 1, & 1 \leq \xi \end{cases}$

So, for $\xi \leq 0$; $u_0(\xi) = 2 \Rightarrow x(t) = 2t + \xi$ are lines with slope $\frac{1}{2}$ emanating from $\xi \leq 0$, and $u = 2$ on them.

For $0 < \xi < 1$; $u_0(\xi) = 2 - \xi \Rightarrow x(t) = (2 - \xi)t + \xi$, lines of slope $\frac{1}{2 - \xi}$ from $0 < \xi < 1$
 $\Rightarrow \xi = \frac{x - 2t}{1 - t} \Rightarrow u(x, t) = 2 - \xi = 2 - \frac{x - 2t}{1 - t} = \frac{2 - x}{1 - t}$

For $1 \leq \xi$; $u_0(\xi) = 1 \Rightarrow x(t) = t + \xi$, lines of slope $\frac{1}{1}$, $u = 1$ on them.



Observe that characteristics intersect at time $t = 1$!!!
 and u becomes discontinuous!
 A shock forms at $t = 1$
 even though the initial data were essentially smooth.

The classical C^1 -smooth solutions only exists up to time $t = 1$!
 What happens afterwards? We need some other concept of solution!

The shock formation time can be found by looking at u_x , to see when it becomes singular;

Along a characteristic $x - ut = \xi$ we have

$$u = u_0(x - ut) \Rightarrow u_x = u_0'(\xi) \cdot [1 - u_x \cdot t] \Rightarrow u_x = \frac{u_0'(\xi)}{1 + u_0'(\xi) \cdot t}$$

which blows up when $1 + u_0'(\xi) \cdot t = 0$, i.e. when

$$t = -\frac{1}{u_0'(\xi)} \text{ which is } > 0 \text{ when } u_0'(\xi) < 0.$$

So a shock will form whenever $u_0'(x)$ is negative somewhere!

$$\text{blow up time } T_b = -\frac{1}{\min_x u_0'(x)}$$

In our example: $u_0'(\xi) = \begin{cases} 0 & \xi \leq 0 \\ -1 & 0 < \xi < 1 \\ 0 & \xi \geq 1 \end{cases} \Rightarrow T_b = -\frac{1}{-1} = 1$ is the shock formation time.

Lesson! Hyperbolic PDEs are non-smoothing; in great contrast to parabolic ones

Non-smooth features (corners, discontinuities in u , u_x , etc) are propagated into the future.

Linear equations propagate discontinuities along characteristics only,

Nonlinear equations may ^{create discontinuities and} propagate them along non-characteristics too (shock curves)

Weak solution formulation

Now we know what we are looking for, ~~but~~ we need to allow discontinuous solutions! Certainly the ^{classical} PDE formulation makes no sense.

But recall that it was obtained from the integral formulation of the conservation law under the assumption of smoothness!

The integral formulation still makes sense:

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x,t) dx = F(u(x_1,t)) - F(u(x_2,t)) \quad \forall x_1, x_2, t$$

or, equivalently, integrating over $[t_1, t_2]$:

$$\int_{x_1}^{x_2} u(x, t_2) dx - \int_{x_1}^{x_2} u(x, t_1) dx = \int_{t_1}^{t_2} F(u)|_{x=x_1} dt - \int_{t_1}^{t_2} F(u)|_{x=x_2} dt$$

$\forall x_1, x_2, t_1, t_2.$

A $u(x,t)$ satisfying this can be thought of as a generalized sense of solution of the PDE, since it can even be discontinuous. Only integrability is needed. This is, of course, the type of solution we learned how to compute with our control-volume advection schemes!

This form is very convenient for discretization, ^{of conservation laws,} as we saw and used, but not convenient for mathematical treatment. Instead, we use the following general test-function approach, which can be used on any PDE.

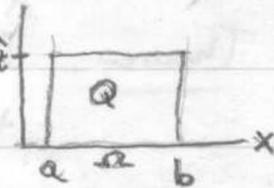
We seek an integral reformulation of the PDE, with no derivatives on u :

Consider a smooth solution of $u_t + F(u)_x = 0$ in $\Omega = [a, b]$, $0 < t < \hat{t}$

Multiply by a smooth test-function $\varphi(x,t)$ (to be chosen ^{appropriately} later)

$$u_t \varphi + F_x \varphi = 0$$

$$(u\varphi)_t - u\varphi_t + (F\varphi)_x - F\varphi_x = 0$$



integrate over $\Omega = [a, b]$ and $[0, \hat{t}]$; i.e. over $Q := \Omega \times (0, \hat{t})$

$$\int_0^{\hat{t}} \int_a^b \{ (u\varphi)_t + (F\varphi)_x \} dx dt - \iint_Q \{ u\varphi_t + F\varphi_x \} dx dt = 0$$

$$\Rightarrow \iint_Q \{ u\varphi_t + F\varphi_x \} dx dt = \int_{\partial Q} u\varphi n_t ds + \int_{\partial Q} F\varphi n_x ds$$

$\vec{n} = (n_x, n_t) = \text{unit normal to } \partial Q$

$$= \underbrace{\int_a^b u\varphi|_{t=0} dx}_{\text{from IC}} + \underbrace{\int_a^b u\varphi|_{t=\hat{t}} dx}_{\text{unknown}} + \underbrace{\int_0^{\hat{t}} F\varphi|_{x=b} dt - \int_0^{\hat{t}} F\varphi|_{x=a} dt}_{\text{from BCs}}$$

Now consider the ^{Cauchy} problem:
$$\begin{cases} u_t + F_x = 0 & \text{in } -\infty < x < \infty, t > 0 \\ u(x, 0) = u_0(x) \end{cases}$$

A smooth solution $u(x, t)$ will satisfy, on any interval (a, b) the integral identity for any smooth $\varphi(x, t)$. The last 3 terms in it are not specified by the CP so we choose $\varphi = 0$ there, and get

$$\int_0^{\hat{t}} \int_a^b \{ u \varphi_t + F \varphi_x \} dx dt + \int_a^b u_0(x) \varphi(x, 0) dx = 0$$

on any interval (a, b) , $\hat{t} > 0$ and any smooth φ from the set

$$\Phi = \{ \varphi \in C^1(\bar{Q}) : \varphi|_{x=a} = \varphi|_{x=b} = \varphi|_{t=\hat{t}} = 0 \} = C_0^\infty(Q)$$

Note that this has no derivatives on u , so in fact it holds for much more general functions $u(x, t)$, only the integrals need to exist:

Definition: We say u is an L^2 -weak solution of the CP, $\begin{cases} u_t + F_x = 0 & \text{in } Q \\ u(x, 0) = u_0(x) \end{cases}$

if $u, F(u) \in L^2(Q)$ and

$$\iint_Q \{ u \varphi_t + F \varphi_x \} dx dt + \int_a^b u_0(x) \varphi(x, 0) dx = 0$$

$$\forall \varphi \in \Phi = \{ \varphi \in C^1(\bar{Q}) : \varphi|_{x=a} = \varphi|_{x=b} = \varphi|_{t=\hat{t}} = 0 \}$$

We saw that a smooth sol. of the CP is an L^2 -weak sol. on any $Q = (a, b) \times (0, \hat{t})$

Conversely, a smooth weak solution will be a classical solution of the CP.

A smooth weak sol is a classical sol, i.e.,

Theorem: If u is an L^2 -weak solution of the CP in $Q = (a, b) \times (0, \hat{t})$ and if $u \in C^1(Q) \cap C(\bar{Q})$ and $F(u)_x \in C(Q)$ then u is ^{also} a classical sol. of the CP.

Proof: Choose $\varphi \in \Phi$, $\varphi(x, 0) = 0$. Then u being a weak sol. $\Rightarrow \iint_Q \{ u \varphi_t + F \varphi_x \} dx dt = 0$

Integrating by parts, all boundary integrals vanish and we get $\iint_Q \varphi \{ u_t + F_x \} dx dt = 0 \quad \forall$ such φ in particular $\forall \varphi \in C_0^\infty(Q)$ which is dense in $L^2(Q)$ $\therefore u_t + F_x = 0$ "almost everywhere" in Q .

But $u_t + F_x$ is cont. in Q $\therefore u_t + F_x = 0 \quad \forall (x, t) \in Q$.

Next, choose any $\varphi \in \Phi$, and integrate by parts. Since $u_t + F_x = 0$ in Q , we get

$$\int_a^b [u_0(x) - u(x, 0)] \varphi(x, 0) dx = 0 \quad \forall \varphi \in \Phi \Rightarrow u(x, 0) = u_0(x) \text{ a.e. in } (a, b)$$

but $u \in C(\bar{Q})$ $\therefore u(x, 0) = u_0(x) \quad \forall x \in (a, b)$. Therefore u satisfies the CP pointwise in Q .