

Lax-Wendroff scheme: for $u_t + vu_x = 0$ and $v > 0$

FD scheme for advective form $u = (v)^T + \dots$

$$u(x, t+\Delta t) = u(x, t) + \Delta t u_x(x, t) + \frac{1}{2} \Delta t^2 u_{xx}(x, t) + \dots$$

$$-vu_x \quad -vu_{xx} = -v(-vu_x) = v^2 u_{xx}$$

$$= u(x, t) - \Delta t v u_x + \frac{1}{2} \Delta t^2 v^2 u_{xx} + \dots$$

use centered differences: $u_x \approx \frac{U_{i+1} - U_{i-1}}{2\Delta x}$, $u_{xx} \approx \frac{U_{i-1} - 2U_i + U_{i+1}}{\Delta x^2}$

$$U_i^{n+1} = U_i^n + \frac{v\Delta t}{2\Delta x} [U_{i+1}^n - U_{i-1}^n] + v^2 \frac{\Delta t^2}{2\Delta x^2} [U_{i-1}^n - 2U_i^n + U_{i+1}^n]$$

$$= U_i^n + \frac{1}{2}\mu [U_{i+1}^n - U_{i-1}^n] + \frac{1}{2}\mu^2 [U_{i-1}^n - 2U_i^n + U_{i+1}^n] = (1-\mu^2)U_i^n + \frac{\mu}{2}(1+4\mu)U_{i-1}^n + \frac{\mu^2-\mu}{2}U_{i+1}^n$$

It is 2nd order, works well on very smooth data only,

but very dispersive and produces oscillations near steep gradients or discontinuities.

Monotonicity: $\mu^2 \geq \mu$ and $1 - \mu^2 \geq 0 \Rightarrow \mu^2 \leq 1$ and $0 < \mu \leq \mu^2 \leq 1 \Rightarrow \mu = 1$ only
 $\mu \leq 0 \leq \mu^2$, $|\mu| \leq 1$ so only for $v < 0$, $|\mu| \leq 1$

The difficult task for advection schemes is to capture discontinuities,

and very steep gradients. Here is how the above 2 methods do:



Upwind ($\mu = \frac{v\Delta t}{\Delta x} \approx \frac{1}{2}$)

shows numerical diffusion

1st order but actually $O(\Delta t^{1/2})$

near discontinuities



Lax-Wendroff ($\mu = \frac{v\Delta t}{\Delta x} \approx \frac{1}{2}$)

less numerical diffusion but has oscillations!

2nd order but actually $O(\Delta t^{2/3})$

near discontinuities

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Approximates to 2nd order the advection-diffusion:

$$u_t + vu_x = \frac{v\Delta x}{2} \left(1 - \frac{v\Delta t}{\Delta x}\right) u_{xx}$$

numerical diffusivity
(artificial viscosity)

Approximates to 3rd order the advection-dispersion eqn:

$$u_t + vu_x = \frac{v\Delta x^2}{6} \left(\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1\right) u_{xxx}$$

Note that taking $\frac{v\Delta t}{\Delta x} = 1$ eliminates the numerical viscosity! But then it is at the verge of instability and may go unstable after many steps

Consistency error and numerical diffusion

For Upwind scheme: $\frac{1}{\Delta t} [U_i^{n+1} - U_i^n] + \frac{v}{\Delta x} [U_i^n - U_{i-1}^n] = 0 \quad (\text{for } v > 0)$

Expand about (x_i, t_n) :

$$U_i^{n+1} = U_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \dots$$

$$U_{i-1}^n = U_i^n - \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} - \frac{\Delta x^3}{6} u_{xxx} + \dots$$

$$\text{FDE}[u] = u_t + \frac{\Delta t}{2} u_{tt} + \dots + v \left[u_x - \frac{\Delta x}{2} u_{xx} + \dots \right]$$

$$\text{but } u_t = -vu_x \Rightarrow u_t = -vu_x = -v(-vu_x)_x = v^2 u_{xx}$$

$$= (u_t + vu_x) + \frac{\Delta t}{2} v^2 u_{xx} - \frac{\Delta x}{2} vu_{xx} + O(\Delta x^2)$$

$\approx \mu = \text{constant number}$

$$= \text{PDE}[u] + \frac{v \Delta x}{2} \left(\frac{v \Delta t}{\Delta x} - 1 \right) u_{xx} + O(\Delta x^2)$$

$$\Rightarrow \text{ter} = \text{FDE}[u] - \text{PDE}[u] = \frac{v \Delta x}{2} (\mu - 1) u_{xx} + O(\Delta x^2) \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

$\text{PDE}[u] =$

Hence, the scheme is consistent and $\approx 1^{\text{st}}$ order accurate for $u_t + vu_x = 0$.

but notice that it is 2^{nd} order accurate for the advection-diffusion equation:

$$-u_t + vu_x = -\frac{v \Delta x}{2} \underbrace{\left(\frac{v \Delta t}{\Delta x} - 1 \right)}_{D_{\text{num}}} u_{xx} = \underbrace{\frac{v \Delta x}{2} \left(1 - \frac{v \Delta t}{\Delta x} \right)}_{D_{\text{num}}} u_{xx}$$

Note that the choice $\mu = \frac{v \Delta t}{\Delta x} = 1$ makes this numerical $D_{\text{num}} = 0$ and thus consistent with the advection eqn. itself again, and of 2^{nd} order, in fact exact: $\mu = 1 \Rightarrow U_i^{n+1} = U_{i-1}^n$!

However, for any $\mu = \frac{v \Delta t}{\Delta x} \leq 1$ (necessary for stability) in fact approximates an advection-diffusion equation, which produces diffused steep profiles:

So Upwinding adds an artificial "numerical diffusion"

$$D_{\text{num}} = \frac{v \Delta x}{2} (1 - \mu), \text{ reduces for finer mesh}$$

and worsens for large v !

Consistency: relation between true value & approximation

For Lax-Wendroff scheme

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{v}{2\Delta x} [U_{i+1}^n - U_i^n] - \frac{v^2 \Delta t}{2\Delta x^2} [U_{i-1}^n - U_i^n + U_{i+1}^n - U_i^n] = 0$$

$$U_i^{n+1} = U_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{3!} u_{ttt} + \frac{\Delta t^4}{4!} u_{tttt} + \dots$$

$$U_{i-1}^n =$$

$$U_{i+1}^n =$$

$$\begin{aligned} FDE[u] = & u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} + \frac{\Delta t^3}{4!} u_{tttt} + \dots \\ & + \frac{v}{2\Delta x} \left[2\Delta x u_x + \frac{2}{6} \Delta x^3 u_{xxx} + 2 \frac{\Delta x^5}{5!} D_x^5 u + \dots \right] \\ & - \frac{v^2 \Delta t}{2\Delta x^2} \left[-\Delta x u_x + \frac{\Delta x^2}{2} u_{xx} - \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{4!} u_{xxxx} - \frac{\Delta x^5}{5!} D_x^5 u + \frac{\Delta x^6}{6!} D_x^6 u + \dots \right. \\ & \quad \left. + \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{4!} u_{xxxx} + \frac{\Delta x^5}{5!} D_x^5 u + \frac{\Delta x^6}{6!} D_x^6 u + \dots \right] \end{aligned}$$

$$\text{but } u_{tt} = v^2 u_{xx}, \quad u_{ttt} = -v^3 u_{xxx}, \quad u_{tttt} = +v^4 u_{xxxx}$$

$$\begin{aligned} &= u_t + \cancel{\frac{v^2 \Delta t}{2} u_{xx}} - \frac{v^3 \Delta t^2}{6} u_{xxx} + \frac{v^4 \Delta t^3}{4!} u_{xxxx} + \dots \\ &\quad + vu_x + \frac{v \Delta x^2}{6} u_{xxx} + \frac{v \Delta x^4}{5!} D_x^5 u + \dots \\ &\quad - \frac{v^2 \Delta t}{2} \left[u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} + \frac{2 \Delta x^4}{6!} D_x^6 u + \dots \right] \end{aligned}$$

$$\begin{aligned} &= (u_t + vu_x) + \frac{v \Delta x^2}{6} \left(1 - \frac{v^2 \Delta t^2}{\Delta x^2} \right) u_{xxx} + \frac{v^2 \Delta x^2 \Delta t}{24} \left(\frac{v^2 \Delta t^2}{\Delta x^2} - 1 \right) u_{xxxx} + \dots \\ &= 0 + \frac{v \Delta x^2}{6} (1 - \mu^2) u_{xxx} + \frac{v \Delta x^3 \mu}{24} (\mu^2 - 1) D_x^4 u + \dots \quad \mu = \frac{v \Delta t}{\Delta x} \end{aligned}$$

so the scheme is a 2nd order approximation to $u_t + vu_x = 0$

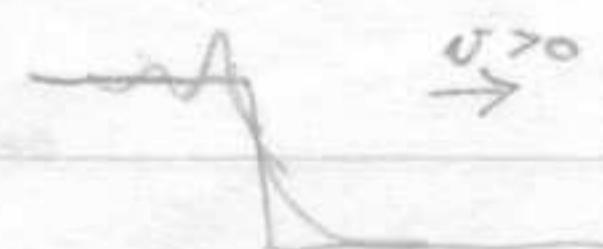
but a 3rd order approximation to the advection-dispersion equation

$$u_t + vu_x = -\frac{v \Delta x^2}{6} \left(1 - \left(\frac{v \Delta t}{\Delta x} \right)^2 \right) u_{xxx} + O(\Delta x^3)$$

The u_{xxx} term is a dispersive one, causes waves to disperse, leading to oscillatory solution,

and since the dispersion coeff. is < 0 , the "group velocity" of waves is $< v$, so the oscillations

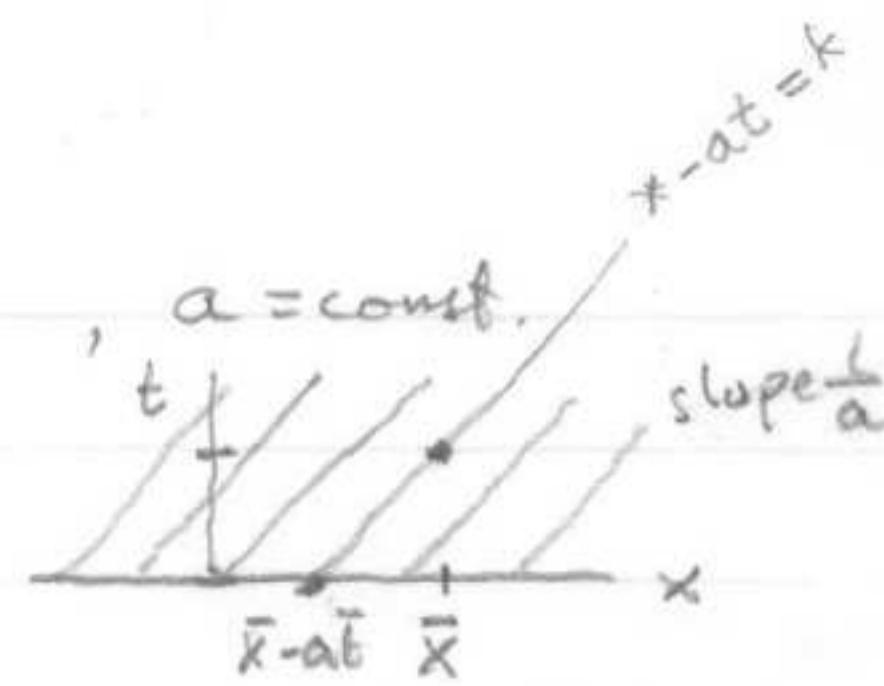
are lagging behind the discontinuity:



(could do normal mode propagation here...
from M535 Lectures...)

Method of characteristics for the linear advection eqn: $u_t + au_x = 0$, $a = \text{const.}$

$$\text{Characteristic curves: } \frac{dt}{1} = \frac{dx}{a} \Rightarrow \frac{dx}{dt} = a \Rightarrow x = at + \xi, \xi = \text{arb. const.}$$

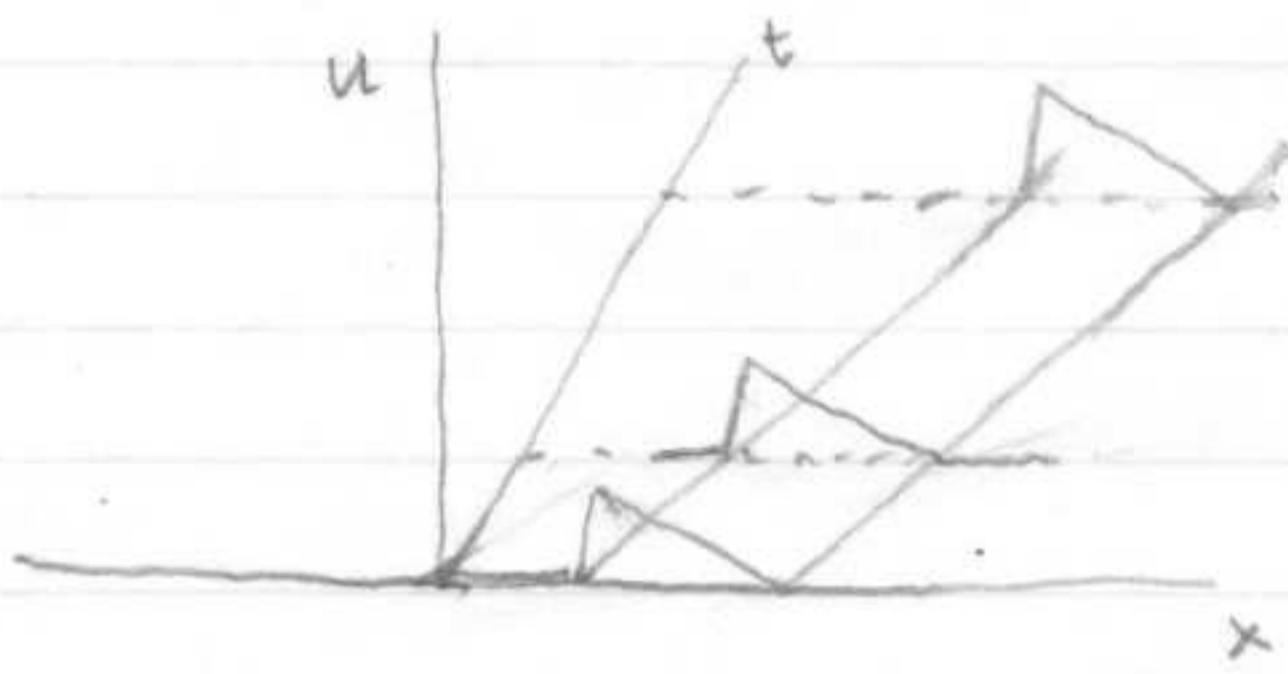


$$\text{Then, along each characteristic: } \frac{du}{dt} = u_t + u_x \frac{dx}{dt} = u_t + au_x = 0 \Rightarrow u = \text{const.}(\xi) = f(x - at)$$

Cauchy IVP: $\begin{cases} u_t + au_x = 0 \\ u(x, 0) = f(x) \end{cases}$ has solution $u(x, t) = f(x - at)$ = undistorted "wave" (shape) traveling right if $a > 0$

left if $a < 0$

The "signal" $f(x)$ propagates along characteristics, with speed a .



Solution at a pt \bar{x} at time \bar{t} : $u(\bar{x}, \bar{t}) = f(\bar{x} - a\bar{t})$ = value of $f(\cdot)$ at where characteristic hits x-axis

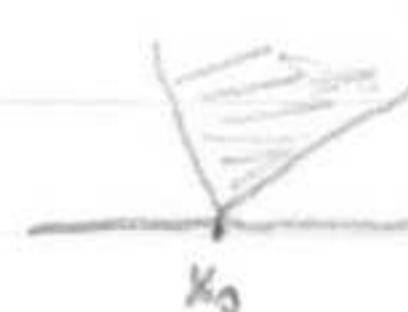
Domain of dependence of $u^{\text{at}}(\bar{x}, \bar{t})$ is the set of initial points on which $u(\bar{x}, \bar{t})$ depends, here $\{\bar{x}_0\}$
 $\bar{x} = a\bar{t}$

For systems of 1st order PDEs, this would be an interval

Range of influence of an initial pt x_0 is the set of (x, t) where $u(x, t)$ is influenced by data at x_0 .

here the char.

For a system, this would be a cone:



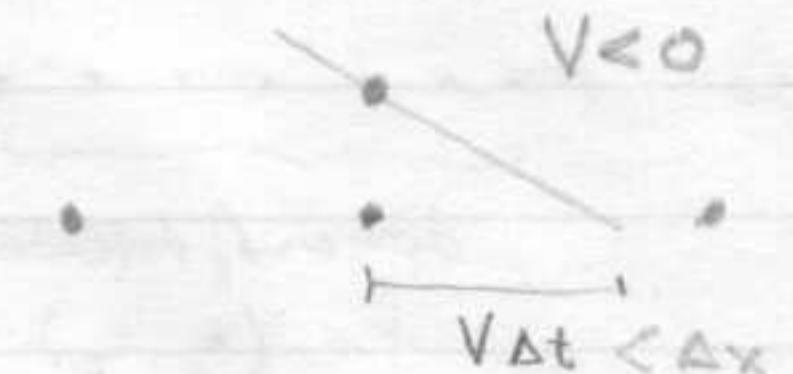
The CFL condition, characteristics, domain of dependence

The exact solution of $U_t + Vu_x = 0$ is $u(x, t) = u_0(x - Vt)$,
so over a single Δt we have

$$u(x_i, t_{n+1}) = u(x_i - V\Delta t, t_n)$$



Tracing the characteristic through (x_i, t_{n+1}) backwards to time t_n :



If $|V|\Delta t > \Delta x$ then (if $|\mu| > 1$) the point $x_i - V\Delta t$ is outside the domain of dependence $[x_{i-1}, x_i]$ or $[x_i, x_{i+1}]$ ($V > 0$) ($V < 0$)

($\mu = \frac{V\Delta t}{\Delta x}$) of the numerical sol., so the num. sol. cannot possibly capture what the exact sol. is doing, so it cannot possibly converge to it as $\Delta x, \Delta t \rightarrow 0$!

We see that the CFL condition: $|\mu| \leq 1$ is necessary for stability and convergence!

If $|V|\Delta t = \Delta x$, i.e. $|\mu| = 1$, then $x_i - V\Delta t = x_{i-1}$ or x_{i+1} , so $U_i^{n+1} = U_{i-1}^n$ or U_{i+1}^n and we obtain the exact sol.? (as you observed in Labs for dt factor = 1)!

If $|V|\Delta t < \Delta x$, i.e. $|\mu| < 1$, then $x_i - V\Delta t$ lies inside $[x_{i-1}, x_i]$ so we can approximate the exact $u(x_i - V\Delta t, t_n)$ by interpolation of the nodal values U_{i-1}^n, U_i^n .

Linear interpolation of the nodal values: $p_1(x) = U_i^n + (x - x_i) \frac{U_i^n - U_{i-1}^n}{\Delta x}$

$$\Rightarrow U_i^{n+1} = p_1(x_i - V\Delta t) = U_i^n - \frac{V\Delta t}{\Delta x} [U_i^n - U_{i-1}^n] \\ = U_i^n + \frac{V\Delta t}{\Delta x} [U_{i-1}^n - U_i^n]$$

which is exactly the Upwind Scheme!

$$= (1-\mu) U_i^n + \mu U_{i-1}^n$$

= convex combination of U_i^n, U_{i-1}^n , ensuring monotonicity, but also smoothing (artificial diffusion)!

Quadratic interpolation of the nodal values $U_{i-1}^n, U_i^n, U_{i+1}^n$:

$$p_2(x) = U_i^n + (x - x_i) \frac{U_i^n - U_{i-1}^n}{\Delta x} + (x - x_i)(x - x_{i+1}) \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{2\Delta x^2}$$

$$\Rightarrow U_i^{n+1} = p_2(x_i - V\Delta t) = U_i^n - \frac{V\Delta t}{\Delta x} [U_i^n - U_{i-1}^n] - \frac{V\Delta t}{2\Delta x^2} (\Delta x - V\Delta t) [U_{i-1}^n - 2U_i^n + U_{i+1}^n]$$

$$= U_i^n + \mu [U_{i-1}^n - U_i^n] - \frac{1}{2}\mu(1-\mu) [U_{i-1}^n - 2U_i^n + U_{i+1}^n]$$

$$= U_i^n + \mu U_{i-1}^n - \mu U_i^n - \frac{1}{2}\mu U_{i-1}^n + \mu U_i^n - \frac{1}{2}\mu U_{i+1}^n + \frac{1}{2}\mu^2 [U_{i-1}^n - 2U_i^n + U_{i+1}^n]$$

$$= U_i^n + \frac{1}{2}\mu [U_{i-1}^n - U_{i+1}^n] + \mu^2 [U_{i-1}^n - 2U_i^n + U_{i+1}^n]$$

The Lax-Wendroff scheme!

Numerical domain of dependence of a grid point (x_j, t_n) is the set of all grid points $(x_i, 0)$ on which U_j^n depends.

t_3
 t_2
 t_1
 $t_0=0$ $\dots \dots \dots$
 domain of dependence
 for (x_j, t_3)
 for any 3-point scheme

As we refine the grid, keeping $\frac{\Delta t}{\Delta x} = \text{fixed}$, the interval of dependence stays the same, but contains twice as many grid points (if we double μ)

As we keep refining, it will fill the interval $[x_j - \frac{T}{r}, x_j + \frac{T}{r}]$ numerical domain of dependence

The exact sol. $u(x_j, T) = u_0(x_j - VT)$, so unless $x_j - VT$ lies inside this interval there is no hope of the num. sol. converging to the exact solution.
Indeed, unless

$$x_j - \frac{T}{r} \leq x_j - VT \leq x_j + \frac{T}{r},$$

the exact sol, which depends only on $u_0(x_j - VT)$, may be anything, and the num. sol. won't know it!

CFL condition: For a num. sol. to be convergent, it is necessary that its num. domain of dependence contain the true domain of dependence of the PDE, as $\Delta x, \Delta t \rightarrow 0$.

Note that these inequalities say $|V| \leq \frac{1}{r} = \frac{\Delta x}{\Delta t}$, i.e. $|V| \frac{\Delta t}{\Delta x} \leq 1$, or $|V| \Delta t \leq \Delta x$, so over a single Δt , the characteristic we trace back must lie within one Δx of x_j .

The CFL condition is only necessary, not sufficient!

The unstable scheme $U_j^{n+1} = U_j^n + \mu \frac{U_{j-1}^n - U_{j+1}^n}{2}$ satisfies it for $|\mu| < 1$ (has same num. domain of dependence as Lax-Wendroff) but it's not convergent since ~~unstable~~

Note that the "positive coeff. rule" for monotonicity is more demanding than the CFL condition:

unstable scheme: $-\frac{\mu}{2} > 0$ impossible \Rightarrow non-monotone but satisfies CFL for $|\mu| \leq 1$.

Lax-Wendroff: $\mu^2 - \mu > 0$ and $1 - \mu^2 \geq 0$

if $\mu > 0$: $\mu \geq 1$ and $\mu^2 \leq 1$ impossible unless $\mu = 1$.

if $\mu < 0$: $\mu^2 \geq 0 \geq 1$ and $|\mu| \leq 1$ OK

so monotone only for $V \leq 0$ provided $|\mu| < 1$, but CFL holds $|\mu| \leq 1$

Heat Eqn. True sol. has domain of depend. all of \mathbb{R}^2 ! By taking $\mu = \frac{\Delta x \Delta t}{\Delta x^2} \leq \frac{1}{2}$, the num. domain of dependence $[x_j - \frac{T}{r}, x_j + \frac{T}{r}]$ has length $\frac{2}{r} = 2 \frac{\Delta x}{\Delta t} \geq \frac{D}{\Delta x} \rightarrow \infty$ as $\Delta x \rightarrow 0$, so CFL cond. is satisfied! as $\Delta x \rightarrow 0$