

Measuring errors in grid functions

(LeVeque 07, FD-ODE-PDE, Appendix A)

Std function norms: $\|u\|_\infty = \max_{x \in \Omega} |u(x)|$, $\|u\|_1 = \int_{\Omega} |u(x)| dx$, $\|u\|_2 = \left(\int_{\Omega} |u(x)|^2 dx \right)^{1/2}$, $\|u\|_p = \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p}$

(eye norm!) $1 \leq p < \infty$

space: $C(\Omega)$ $L_1(\Omega)$ $L_2(\Omega)$ $L^p(\Omega)$

Some num. methods produce num. solutions that are functions $U(x) \approx u(x) = \text{exact}$
(FE, collocation, spectral, interpolation, ...)

error $= U(x) - u(x) \Rightarrow$ magnitude = $\|\text{error}\|$ in some function norm.

Other methods, like FV, FD, produce only discrete values $U_i \approx u(x_i)$ on a grid $\{x_i\}_{i=1,2,\dots,M}$

$$\text{or } U_i \approx \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x) dx$$

of spacing $h = \Delta x, \Delta t, \dots = O(\frac{1}{M})$.

We call $\{U_1, U_2, \dots, U_M\}$ a grid function and think of it as "discretization" or

"sampling" of the exact $u(x)$ on the grid $\{x_i\}$.

$\text{error}_i = U_i - u(x_i)$ is also a grid function on the same grid.

or averages

magnitude of error: in max norm (eye norm) $= \|\vec{e}\|_\infty = \max_i |e_i|$ same as vector $\| \cdot \|_\infty$

but in any of the integral norms we must use the average, not just the sum!

$$\|\vec{e}\|_1 = \frac{1}{M} \sum_{i=1}^M |e_i| = h \cdot \sum_i |e_i| \quad (\text{not } \sum_{i=1}^M |e_i| \text{ which depends on } M = \text{grid size!})$$

$$\|\vec{e}\|_q = \left(h \sum_{i=1}^M |e_i|^q \right)^{1/q}, \quad 1 \leq q < \infty$$

$$\text{Similarly in 2D, 3D, ...: } \|\vec{e}\| = \left(Ax \cdot Ay \sum_i \sum_j |e_{ij}|^q \right)^{1/q}, \quad \text{etc.}$$

Norm equivalence is lost for grid functions when we refine grids!

It holds in each finite dimil space \mathbb{R}^M , but as we refine $h \rightarrow 0$ we increase $M = \frac{b-a}{h}$, and the norm equivalence constants depend on h, M :

$$h \|\vec{e}\|_\infty \leq \|\vec{e}\|_1 \leq hM \|\vec{e}\|_\infty$$

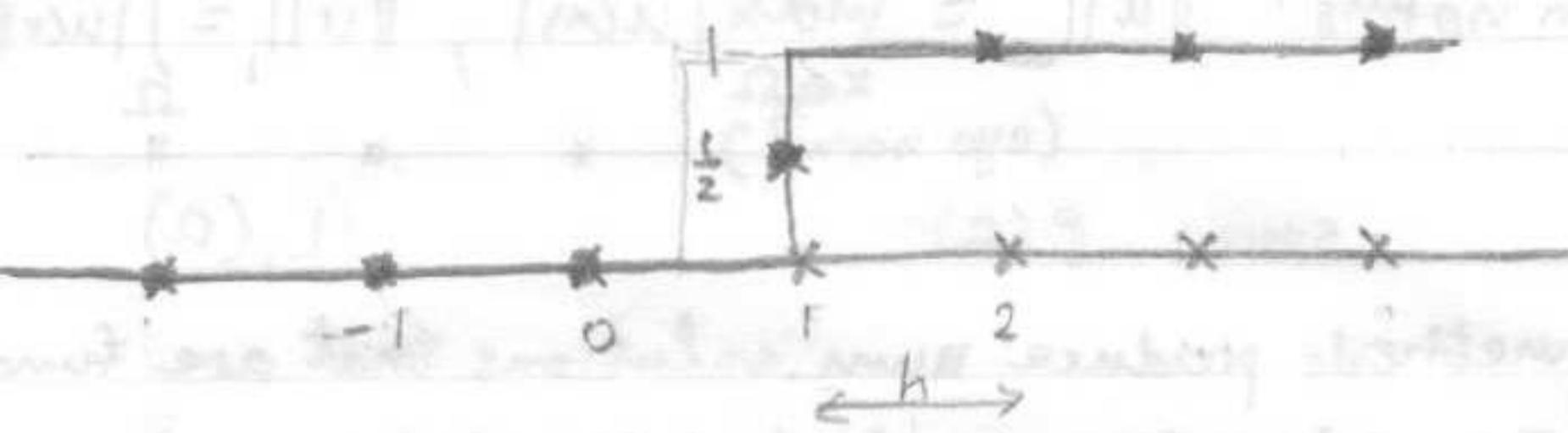
$$\sqrt{h} \|\vec{e}\|_\infty \leq \|\vec{e}\|_2 \leq \sqrt{hM} \|\vec{e}\|_\infty$$

$$\sqrt{h} \|\vec{e}\|_2 \leq \|\vec{e}\|_1 \leq \sqrt{hM} \|\vec{e}\|_2$$

For discontinuous functions, breakdown of norm equivalence is striking,
which norm we use makes a big difference!

Example: $u(x) = \begin{cases} 0, & x \leq \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$

U_i as in the figure



Then $e_i = \begin{cases} 0, & x_i \neq \frac{1}{2} \\ \frac{1}{2}, & x_i = \frac{1}{2} \text{ (at single grid pt)} \end{cases} \Rightarrow \|\vec{e}\|_\infty = \frac{1}{2} \text{ for all } h, \rightarrow 0 \text{ as } h \rightarrow 0$

But $\|\vec{e}\|_1 = h \cdot \frac{1}{2} = O(h) \rightarrow 0 \text{ as } h \rightarrow 0$

So, this grid function U_i does not approximate u well in ∞ -norm,
but it does in 1-norm. Good or bad depends on what we are looking for...

Moreover, if we define $e_i = U_i - \frac{1}{h} \int_{x_i - \frac{h}{2}}^{x_i + \frac{h}{2}} u(x) dx$ then $e_i(h) = 0 \forall i$

so now $\|\vec{e}\| = 0$ in any norm, even in max-norm!

This is the appropriate concept of error for approximation of discontinuous $u(x)$.

Want both fair for operators etc from one frame so must drop the inf norm in 1-d

$$\|f\|_M = \max_{x \in [0,1]} |f(x)| \text{ for } f(x) = 1.913 \cdot d \approx 1.913 \cdot \frac{1}{M} = \|f\|_1$$

$$\Rightarrow \|f\|_M = \left(1.913 \cdot \frac{1}{M}\right)^2 = \|f\|_2^2$$

$$\|f\|_M = \|f\|_2 \quad \text{as } M \rightarrow \infty$$

$$\|f\|_M \geq \|f\|_1 \geq \|f\|_2$$

$$\|f\|_M \geq \|f\|_1 \geq \|f\|_2$$

$$\|f\|_M \geq \|f\|_1 = \|f\|_2$$

From LeVeque, Finite Diff Methods for DEs, 1998, Appendix pp A10-12

Estimating order of convergence from grid functions on exact sol.

When we apply a method of order p : $\| \text{error} \| = O(h^p)$ to a problem,

we need to debug on a problem whose exact sol we know, so we can evaluate E_i .

Let $\vec{U}(h) = \text{num. sol. (grid function)} \text{ on grid with spacing } h$

$\vec{u}(h) = \text{exact sol. evaluated on same grid}$

error: $E(h) = \| \vec{U}(h) - \vec{u}(h) \|$ w.r.t. some norm

If method is of order p , we expect $E(h) = Ch^p + o(h^p)$ as $h \rightarrow 0$

so for small enough h : $E(h) \approx Ch^p$

If we refine grid by factor of 2, say, we expect $E(\frac{h}{2}) \approx C \left(\frac{h}{2}\right)^p$

error ratio $R(h) = \frac{E(h)}{E(\frac{h}{2})}$, we expect $R(h) \approx 2^p$

therefore $p \approx \log_2(R(h)) = \log_2 \frac{E(h)}{E(\frac{h}{2})}$: can be found from two runs,
one with h and one with $\frac{h}{2}$, h small enough

This holds for any refinement ratio $\frac{h_1}{h_2}$, $p \approx \frac{\log \left(\frac{E(h_1)}{E(h_2)} \right)}{\log \left(\frac{h_1}{h_2} \right)}$

Can also estimate $C \approx \frac{E(h)}{h^p}$ once p is known.

Estimating error without exact solution

After debugging on an exactly solvable problem, we run our code on our real problem

whose solution is not known. How do we know how we are doing?

Compute with finer meshes: $U(h)$, $U(\frac{h}{2})$, $U(\frac{h}{4})$, ... till too slow to compute!
or "values" do not change

1. Using fine-grid solution as reference

If we can afford to run on a very fine grid, of spacing $h_f^{< h}$, want to use it as "exact" sol.

To compare $U(h)$ with $U(h_f)$ need h_f to be a refinement of h , so the h_f grid contains

the coarse h -grid to obtain the restriction $\hat{U}(h)$ on h -grid of the fine $U(h_f)$.

This is not simple to do! big pain getting indexing correct... Else must use interpolation but to order higher than p !

Then $U(h) - \hat{U}(h) = U(h) - u(h) + \underbrace{u(h) - \hat{U}(h)}$

$$\sim Ch_f^p \ll Ch^p \text{ for } p\text{-order method}$$

negligible

so we can take $E(h) \approx U(h) - \hat{U}(h)$ as the "actual" error, and can estimate p

Caution: such grid refinement study is necessary but not sufficient for verification!

It only confirms the code is converging to some function at order p , not necessarily to the true solution!

Should also check the solution makes sense physically, has expected behavior, satisfies the ICs, BCs, ...

To estimate the order p : $E(h) \approx U(h) - \hat{U}(h) \approx Ch^p$, $E(\frac{h}{2}) \approx U(\frac{h}{2}) - \hat{U}(\frac{h}{2}) \approx C(\frac{h}{2})^p$

$$\Rightarrow \text{ratio} \approx 2^p \Rightarrow p = \log_2 \frac{E(h)}{E(\frac{h}{2})}, \quad C \approx \frac{E(h)}{h^p}$$

→ And so does the error go down?

2. Without a fine grid sol: if we can't afford a very fine grid

but only $U(h), U(\frac{h}{2}), U(\frac{h}{4})$, say,

Treat $\bar{h} = \frac{h}{4}$ as "exact":

$$E(h) \approx U(h) - U\left(\frac{h}{2}\right), \quad E\left(\frac{h}{2}\right) \approx U\left(\frac{h}{2}\right) - U\left(\frac{h}{4}\right)$$

$$\approx Ch^p$$

$$\approx C\left(\frac{h}{2}\right)^p$$

$$h = 4\bar{h}, \quad \frac{h}{2} = 2\bar{h}$$

$$\tilde{E}(h) = E(h) - E(\bar{h}) \approx Ch^p - C\bar{h}^p, \quad \tilde{E}\left(\frac{h}{2}\right) = E\left(\frac{h}{2}\right) - E(\bar{h}) \approx C2^p\bar{h}^p - C\bar{h}^p$$

$$\approx (4^p - 1)C\bar{h}^p$$

$$\approx (2^p - 1)C\bar{h}^p$$

$$\Rightarrow \tilde{R}(h) = \frac{\tilde{E}(h)}{\tilde{E}\left(\frac{h}{2}\right)} \approx \frac{4^p - 1}{2^p - 1} = 2^p + 1 \quad (\text{not } 2^p)!$$

so for a 1st order method ($p=1$) we have $\tilde{R}(h) \approx 3$, error should reduce by factor of 3, not 2

$$\Rightarrow p \approx \log_2(\tilde{R}(h) - 1)$$