

Fourier analysis of the Upwind scheme for $u_t + Vu_x = 0$

$$\text{Upwind scheme: } U_j^{n+1} = U_j^n + \mu [U_{j-1}^n - U_j^n] , \quad \mu = \frac{V\Delta t}{\Delta x} , \quad V > 0$$

Consider the evolution of an initial harmonic

$$x_j = j\Delta x$$

$$U_j^0 = e^{ikx_j} = e^{ikj\Delta x} , \quad j=1, 2, \dots, M , \quad -\infty < k < \infty$$

$$\text{We seek normal modes, in the form } U_j^n = \lambda^n e^{ikj\Delta x} , \quad n=0, 1, \dots \\ = \lambda^n U_j^0$$

$$\text{PDE} \Rightarrow \lambda^{n+1} U_j^n = (1-\mu) \lambda^n U_j^n + \mu \lambda^n U_j^0 e^{-ik\Delta x}$$

$$\Rightarrow \lambda(k) = (1-\mu) + \mu e^{-ik\Delta x} \quad \text{is the dispersion relation}$$

$$= [(1-\mu) + \mu \cos k\Delta x] - i\mu \sin k\Delta x \quad \text{= amplification factor per time step}$$

$$\Rightarrow |\lambda(k)|^2 = [\dots]^2 + \mu^2 \sin^2 k\Delta x$$

$$= 1 - 2\mu(1-\mu)(1-\cos k\Delta x)$$

$$= 1 - 4\mu(1-\mu)\sin^2 \frac{k\Delta x}{2}$$

(for $V > 0$)

We have $|\lambda(k)|^2 \leq 1 \quad \forall k \in \mathbb{R}$ provided $0 \leq \mu \leq 1$, the familiar CFL condition again, for stability.

For $0 < \mu < 1$, all modes are damped ($|\lambda(k)| < 1$) so amplitude \downarrow in time, as we saw before.

The (complex) amplification factor $\lambda(k)$ also has a phase ($z = |\lambda| e^{i\theta}$)

$$\arg \lambda(k) = -\arctan \left[\frac{\mu \sin k\Delta x}{(1-\mu) + \mu \cos k\Delta x} \right]$$

We are interested in small $\xi = k\Delta x$ (these are modes that can be approximated well on the mesh).

It can be shown [Morton - Mayers, p. 95] that for small $k\Delta x$

$$\text{relative phase error: } \frac{\arg \lambda(k)}{-\mu \xi} \approx -\mu \xi \left[1 - \frac{1}{6}(1-\mu)(1-2\mu)\xi^2 + \dots \right] , \quad \xi = k\Delta x , \quad \text{for } k\Delta x \rightarrow 0$$

Over a time step, the phase of the exact sol. changes by $-kVBt = -\mu \xi$, so $\mu = 1$ and $\mu = \frac{1}{2} \Rightarrow$ exact phase. Otherwise the phase error is $O(\xi^2)$.

Fourier

Normal mode analysis of advection-diffusion: $u_t + Vu_x = Du_{xx}$

Set $u = u_k(x, t) = a(k) e^{i(kx - \omega t)}$, $u(x, 0) = u_k(x, 0) = a(k) e^{ikx}$

$$u_t = (-i\omega) u_k, \quad u_x = (ik) u_k, \quad u_{xx} = (ik)^2 u_k = -k^2 u_k$$

$$\text{PDE} \Rightarrow -i\omega + V ik = -D k^2 \Rightarrow \omega(k) = -\frac{D k^2}{-i} - \frac{V ik}{-i} = -i D k^2 + V k$$

$$\Rightarrow \text{Im } \omega(k) = -D k^2 \text{ and normal modes are } u_k(x, t) = a(k) e^{-D k^2 t} \cdot e^{ik(x - Vt)}$$

If $D > 0$ (advection-diffusion diffusion) then $\Omega = \sup_{-\infty < k < \infty} (-D k^2) = 0$

\Rightarrow Cauchy problem is well-posed and neutrally stable.

Normal modes are harmonic waves of speed V (driven by the advection term Vu_x) with exponentially decaying amplitude (due to diffusion term $D u_{xx}$).

The larger the diffusivity $D > 0$, the faster the decay,

and modes of small wavelength $\lambda = \frac{2\pi}{k}$ decay faster.

All modes $\rightarrow 0$ as $t \rightarrow \infty$, so PDE is strictly stable.

If $D = 0$ (pure advection) then $\Omega = 0$ again,

Normal modes do not decay, only propagate with speed V ,

If $D < 0$ (backward diffusion) then $\Omega = +\infty$, so Cauchy problem is ill-posed.

Normal modes grow exponentially in time!

The past cannot be determined!

Fundamental properties of

hyperbolic PDEs :- non-smoothing : features propagate with finite speed (along characteristics, which are real)
- forward and backward well posed
- no max principle in space (space waves can be solutions)
- but positivity preserving (monotone in time): $u(x, 0) \geq 0 \Rightarrow u(x, t) \geq 0$ for $t > 0$

parabolic PDEs: - smoothing: rough initial data are smoothed out instantaneously! certain loss of information
- infinite speed of propagation of signals (characteristics have ∞ slope, horizontal)
- time irreversible: only forward in time problems can be well-posed
 backward in time is ill-posed and unstable (cannot recover the past)
- max principle holds in space and time (max, min occur on "parabolic boundary": on $\partial\Omega$ or earlier)
- positivity preserving