

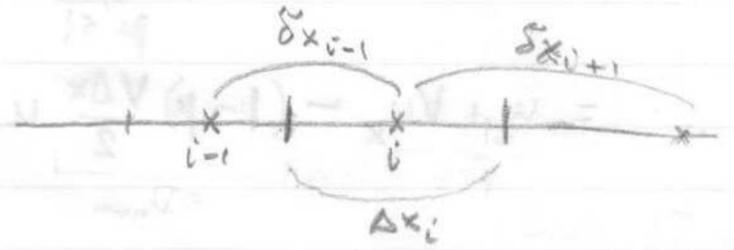
# Consistency analysis of upwind scheme for advection

$u_t + (v u)_x = 0$  i.e.  $u_t + F_x = 0$ ,  $F = v u$ , assume  $v(x,t)$  smooth

upwind scheme:  $U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x_i} [F_{i-1/2}^n - F_{i+1/2}^n]$

Consider  $v > 0$

$F_{i-1/2}^n = v_{i-1/2} U_{i-1}^n$



FDE[U] =  $\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{1}{\Delta x_i} [v_{i+1/2} U_i^n - v_{i-1/2} U_{i-1}^n]$

Taylor in t about  $t_n$ :  $u(x_i, t_{n+1}) = u(x_i, t_n) + \Delta t \cdot u_t(x_i, t_n) + \frac{\Delta t^2}{2!} u_{tt}(x_i, t_n) + O(\Delta t^3)$

Taylor in x about  $x_i$ :  $u(x_{i-1}, t_n) = u(x_i, t_n) - \delta x_{i-1} u_x(x_i, t_n) + \frac{\delta x_{i-1}^2}{2} u_{xx}(\cdot) - \frac{\delta x_{i-1}^3}{3!} u_{xxx}(\cdot) + \dots$

Consistency error  $\rho_n = FDE[u] - (-PDE[u])$

FDE[u] =  $u_t(x_i, t_n) + \frac{\Delta t}{2} u_{tt}(\cdot) + O(\Delta t^2) + \frac{1}{\Delta x_i} [v_{i+1/2} u(x_i, t_n) - v_{i-1/2} (u - \delta x_{i-1} u_x + \frac{\delta x_{i-1}^2}{2} u_{xx} - \frac{\delta x_{i-1}^3}{3!} u_{xxx} + \dots)]$

=  $u_t(x_i, t_n) + \frac{\Delta t}{2} u_{tt} + \frac{v_{i+1/2} - v_{i-1/2}}{\Delta x_i} u + v_{i-1/2} [\frac{\delta x_{i-1}}{\Delta x_i} u_x - \frac{\delta x_{i-1}^2}{2 \Delta x_i} u_{xx} + \frac{\delta x_{i-1}^3}{3! \Delta x_i} u_{xxx} + \dots]$

=  $u_t + \frac{\Delta t}{2} u_{tt} + v_x u + [\frac{\Delta x_i^2}{24} v_{xx} + \dots] u + v_{i-1/2} [\frac{\delta x_{i-1}}{\Delta x_i} u_x - \frac{\delta x_{i-1}^2}{2 \Delta x_i} u_{xx} + \dots]$

but  $v_x u = (v u)_x - v u_x$

=  $[u_t + (v u)_x] - v u_x + \frac{\Delta t}{2} u_{tt} + v_{i-1/2} [\frac{\delta x_{i-1}}{\Delta x_i} u_x - \frac{\delta x_{i-1}^2}{2 \Delta x_i} u_{xx} + \dots] + u [\frac{\Delta x_i^2}{24} v_{xx} + \dots]$

For  $v \equiv \text{const} = V$ :  $u_t + V u_x = 0 \Rightarrow u_{tt} = (-V u_x)_t = -V (u_t)_x = -V (-V u_x)_x = +V^2 u_{xx}$

FDE[u] =  $u_t + V u_x + V [-1 + \frac{\delta x_{i-1}}{\Delta x_i}] u_x + [\frac{\Delta t}{2} V^2 - V \frac{\delta x_{i-1}^2}{2 \Delta x_i}] u_{xx} + O(\Delta t^2) + O(\frac{\delta x_{i-1}^3}{\Delta x_i})$

=  $u_t + \frac{\delta x_{i-1}}{\Delta x_i} \cdot V u_x + \frac{V}{2} (V \Delta t - \frac{\delta x_{i-1}^2}{\Delta x_i}) u_{xx} + \dots$

our PDE with altered  $\alpha$

For uniform mesh:

FDE[u] =  $u_t + V u_x + 0 + \frac{V}{2} (V \Delta t - \Delta x) u_{xx} + O(\Delta t^2) + O(\Delta x^2)$

Consistency analysis of upwind scheme for advection

$$p = u_t + Vu_x + \frac{V\Delta x}{2} \left( \frac{V\Delta t}{\Delta x} - 1 \right) u_{xx} + O(\Delta t^2) + O(\Delta x^2)$$

$$= \underbrace{u_t + Vu_x}_{\text{PDE}} - \underbrace{(1-\mu) \frac{V\Delta x}{2}}_{= D_{\text{num}}} u_{xx} + O(\Delta t^2) + O(\Delta x^2)$$

consistent to 1<sup>st</sup> order

So, the upwind scheme approximates the advection eqn.  $u_t + Vu_x = 0$  to  $O(\Delta x)$ : 1<sup>st</sup> order in  $\Delta x$   
 and " " advection-diffusion eqn.  $u_t + Vu_x - D_{\text{num}} u_{xx} = 0$  to 2<sup>nd</sup> order

with  $D_{\text{num}} = (1-\mu) \frac{V\Delta x}{2}$ ,  $\mu = \frac{V\Delta t}{\Delta x}$

If  $\mu = 1$  then exact to 2<sup>nd</sup> order!

If  $\mu > 1$ , approximates a backward diffusion (ill posed!), unstable

If  $\mu \neq 1$ , approximates diffusion-advection with  $D_{\text{num}} = (1-\mu) \frac{V\Delta x}{2}$   
 still consistent with advection since  $D_{\text{num}} \rightarrow 0$  as  $\Delta x \rightarrow 0$

but diffusive with "numerical" (artificial) diffusion  $D_{\text{num}}$ !

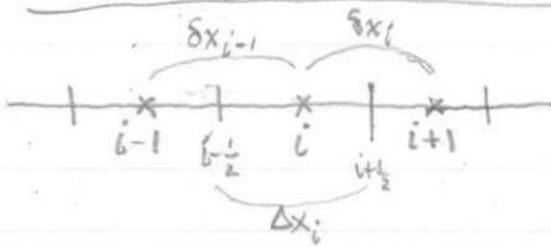
Clearly we should choose  $\mu$  as near 1 as possible (roundoff may make it unstable)

The coarser the mesh and the bigger  $V$  is, the more diffusive it will be!

# Consistency analysis of FV scheme for diffusion

the simplest to do of consistency-stability-convergence, and very informative, get order of scheme. Assume  $u(x,t)$  is smooth.

Consistency of FV scheme for  $u_t = D u_{xx}$  for non-uniform mesh



FV scheme: 
$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{\Delta x_i} [F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}], \quad i=1, 2, \dots, M$$

$$F_{i-\frac{1}{2}} = -D \frac{U_i - U_{i-1}}{\delta x_{i-1}}, \quad i=2, \dots, M \quad (\text{internal faces})$$

Taylor expansion in  $t$ :

$$u(x_i, t_n + \Delta t) = u(x_i, t_n) + \Delta t \cdot u_t(x_i, t_n) + \frac{\Delta t^2}{2} u_{tt}(x_i, t_n) + \mathcal{O}(\Delta t^3)$$

$$\Rightarrow \frac{U_i^{n+1} - U_i^n}{\Delta t} = u_t(x_i, t_n) + \frac{\Delta t}{2} u_{tt}(x_i, t_n) + \frac{\Delta t^2}{3!} u_{ttt} + \dots$$

Taylor in  $x$ : 
$$u(x_{i-1}, t_n) = u(x_i, t_n) - \delta x_{i-1} \cdot u_x(x_i, t_n) + \frac{\delta x_{i-1}^2}{2} u_{xx}(x_i, t_n) - \frac{\delta x_{i-1}^3}{3!} u_{xxx}(x_i, t_n) + \frac{\delta x_{i-1}^4}{4!} u_{xxxx}(x_i, t_n) + \dots$$

$$u(x_{i+1}, t_n) = u(x_i, t_n) + \delta x_i \cdot u_x(x_i, t_n) + \frac{\delta x_i^2}{2} u_{xx}(x_i, t_n) + \frac{\delta x_i^3}{3!} u_{xxx}(x_i, t_n) + \frac{\delta x_i^4}{4!} u_{xxxx}(x_i, t_n) + \dots$$

$$\Rightarrow F_{i-\frac{1}{2}}^n = -D \left[ u_x - \frac{\delta x_{i-1}}{2} u_{xx} + \frac{\delta x_{i-1}^2}{3!} u_{xxx} + \mathcal{O}(\delta x_{i-1}^3) \right]$$

$$F_{i+\frac{1}{2}}^n = -D \left[ u_x + \frac{\delta x_i}{2} u_{xx} + \frac{\delta x_i^2}{3!} u_{xxx} + \mathcal{O}(\delta x_i^3) \right]$$

$$\therefore \text{FDE}[u] = u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} + \dots - \frac{D}{\Delta x_i} \left[ \frac{\delta x_{i-1} + \delta x_i}{2} u_{xx} - \frac{\delta x_{i-1}^2 - \delta x_i^2}{3!} u_{xxx} + \frac{\delta x_{i-1}^3 + \delta x_i^3}{4!} u_{xxxx} + \dots \right]$$

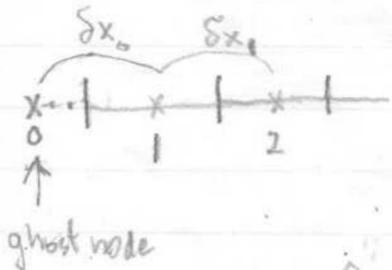
$$\text{PDE} = u_t - \left( \frac{\delta x_{i-1} + \delta x_i}{2 \Delta x_i} \right) D u_{xx} + \frac{\Delta t}{2} u_{tt} - \frac{\delta x_{i-1}^2 - \delta x_i^2}{6 \Delta x_i} D u_{xxx} + \frac{\delta x_{i-1}^3 + \delta x_i^3}{4! \Delta x_i} D u_{xxxx} + \dots$$

For uniform mesh: 
$$\text{FDE}[u] = \underbrace{u_t - D u_{xx}}_{\text{PDE}} + \frac{D \Delta t}{2} u_{tt} + \mathcal{O}(\Delta t^2) u_{ttt} + \mathcal{O}\left(\frac{\Delta x^3}{\Delta x}\right) u_{xxx} + \dots$$
 Note: no consistency error when  $p < \frac{1}{2}$  (no artificial diffusion)

$$\Rightarrow \text{LTE} : \rho_n = \text{FDE}[u] - \text{PDE}[u] = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \therefore \text{the scheme is } 1^{\text{st}} \text{ order in time}$$
  

$$\rightarrow 0 \text{ as } \Delta t, \Delta x \rightarrow 0 \therefore \text{consistent } 2^{\text{nd}} \text{ order in space}$$
  
 but  $\Delta t \sim \Delta x^2$ , so  $\mathcal{O}(\Delta x^2)$

At boundary node: 
$$\text{FDE}[u] = u_t - \frac{\delta x_0 + \delta x_1}{2 \Delta x_1} D u_{xx} + \mathcal{O}(\Delta t, \Delta x^2)$$



want coeff = 1, won't be with  $\delta x_0 = \frac{\Delta x_1}{2}$ ;  $\frac{1}{2} + 1 = \frac{3}{2}$

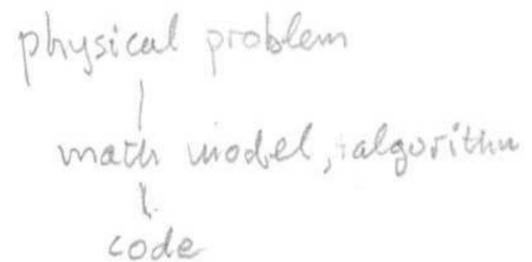
can consider a "ghost node"  $x_0$  as sitting at  $-\frac{\delta x_1}{2}$  so that  $\delta x_0 = \delta x_1$

Or, can use the mesh: 
$$\begin{array}{c} U_0 \quad U_1 \\ x_0 \quad x_1 \end{array} \dots$$
 and conserve only on control volumes again

more convenient for Dirichlet BC's

## Verification and Validation

aspects of quality control, necessary but not sufficient



Verification: the code actually solves the algorithm being implemented for the math model

answers the question: "are you building it right?"  
doing

Validation: the math model (and code) models the actual physical problem of interest

"are you building the right thing?"  
doing

Very difficult to do, predictions from model (code) must "agree" with observations/measurements  
but this only improves our confidence in the model...

Verification involves

1. debugging on an exactly solvable math problem, so can compare numerical & exact solutions  
(ideally debug each module/function, then all together)
2. grid convergence: as  $h \rightarrow 0$ , num. sol. tends to true sol. (errors decrease)  
verify expected order of convergence  
But this does not check correctness, may be producing wrong sol! (e.g. wrong BC), so always:
3. check expected physical behavior, especially on (physical) quantities of interest