

Linear Advection equation

$$\begin{cases} u_t + v u_x = 0, \\ u(x, 0) = u_0(x) \end{cases}, \quad v = \text{const.}, \quad -\infty < x < \infty$$

(initial values)

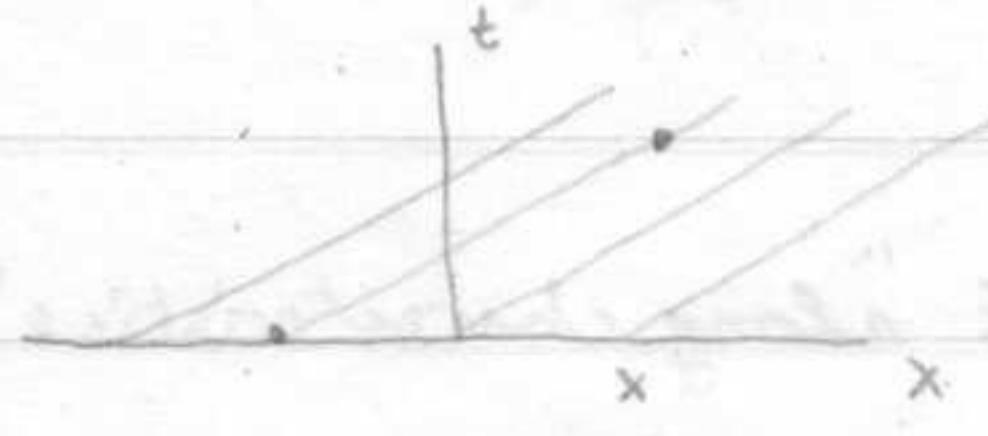
Solution: Find the characteristics:

$$\frac{dt}{ds} = 1 \Rightarrow s = t$$

$$\frac{dx}{dt} = v \Rightarrow x = vt + \xi \quad (\text{straight lines})$$

Along a characteristic $\frac{du}{dt} = 0 \Rightarrow u = k(\xi)$ (arbitrary function of ξ)

$$\Rightarrow u(x, t) = k(x - vt)$$



at $t=0$

$$\text{At } t=0: u(x, 0) = u_0(x) = k(x) \Rightarrow k(x) = u_0(x)$$

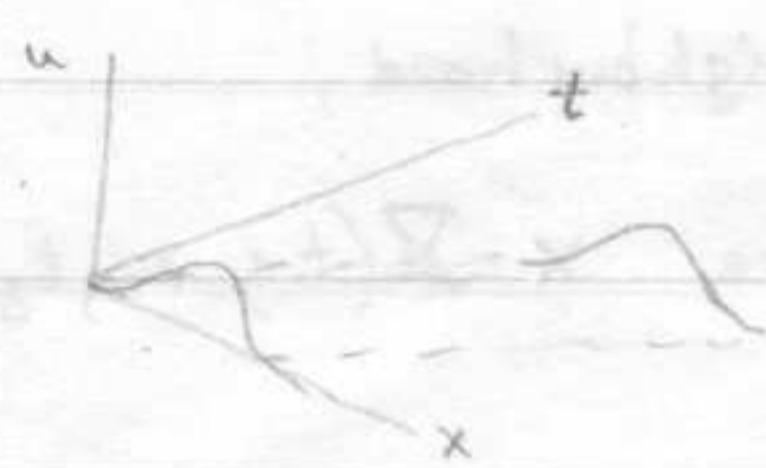
$$\text{Therefore } u(x, t) = u_0(x - vt)$$



This is a traveling wave, the initial profile $u_0(x)$ moving to the right (if $v > 0$) with speed v , or

to the left (if $v < 0$) " "

i.e. moving downstream, undistorted!

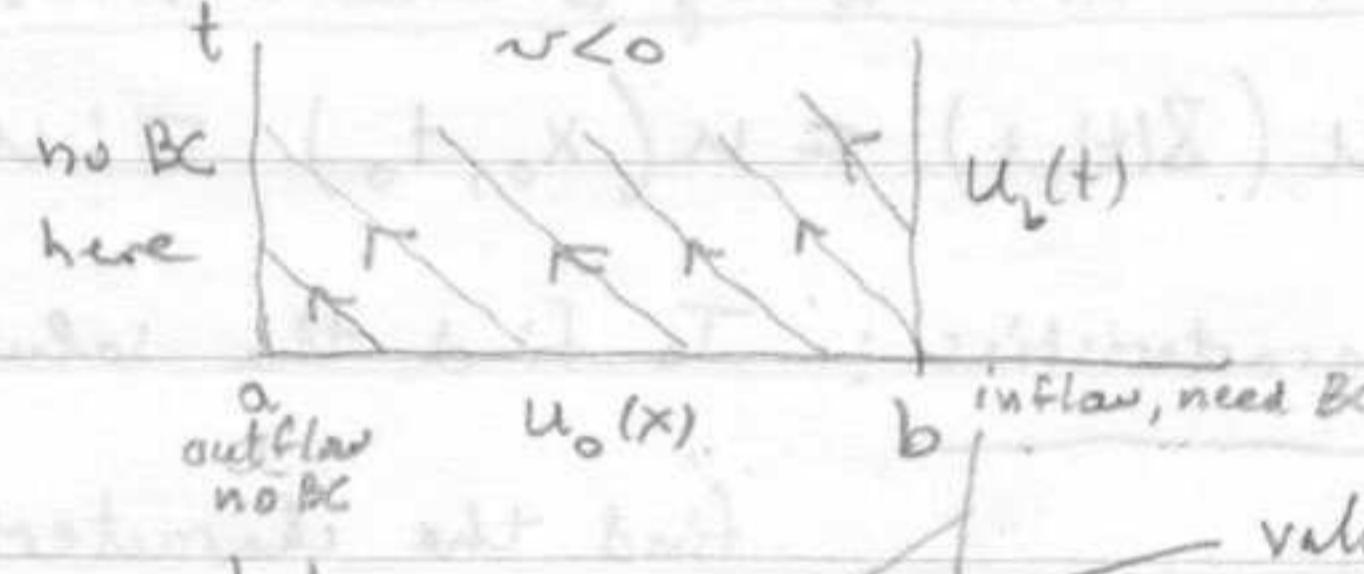
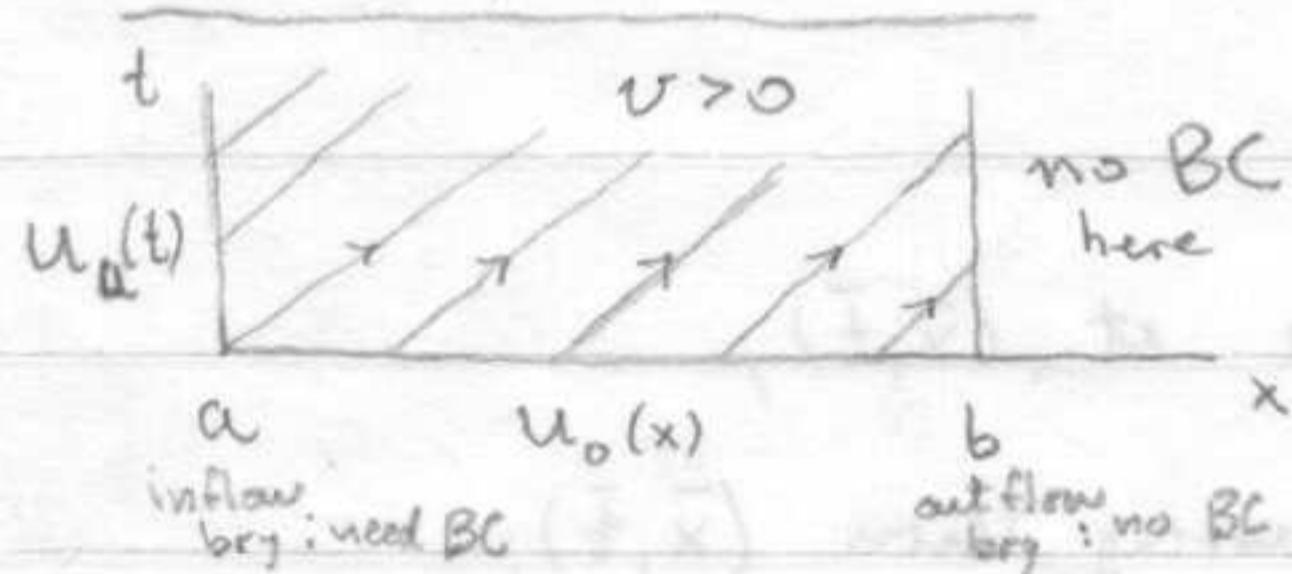


In fact, the advection eqn.

$u_t + vu_x = 0$ is the equation satisfied by a wave traveling undistorted with speed v . Indeed, such a wave must have the form $u(x, t) = f(x - vt)$

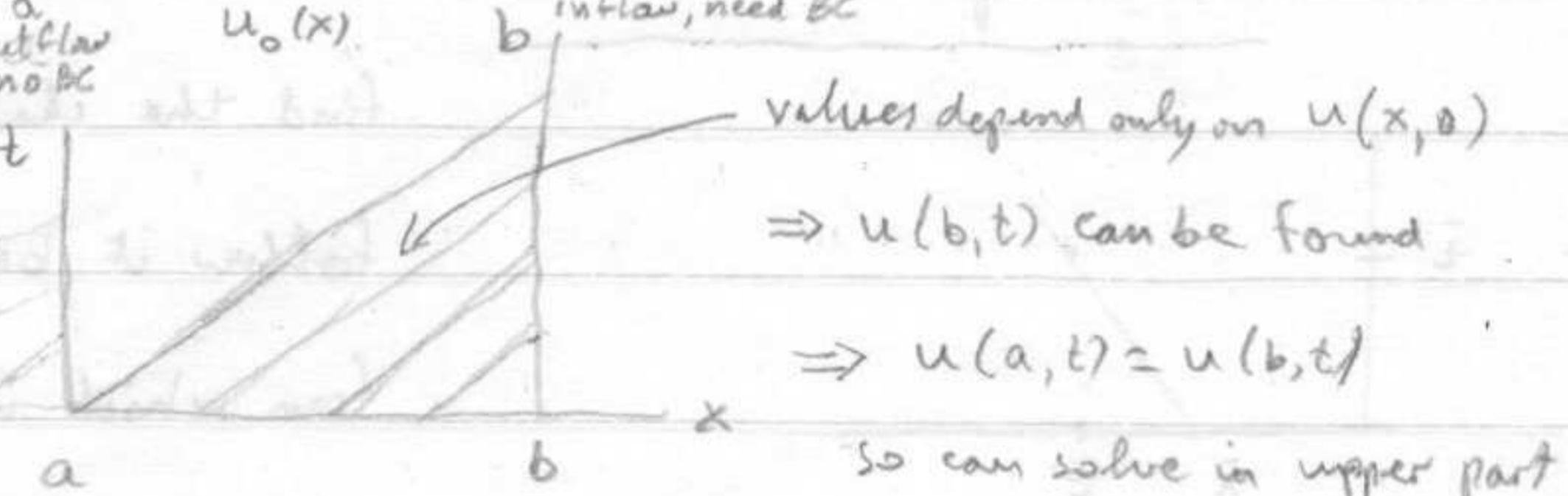
which satisfies: $u_t = f' \cdot (-v)$, $u_x = f'$ $\Rightarrow u_t = u_x \cdot (-v) = -vu_x \Rightarrow u_t + vu_x = 0$.

Boundary conditions: The characteristics tell us where we need them:



values depend only on $u(x, 0)$

Periodic BC: $u(a, t) = u(b, t)$



$\Rightarrow u(b, t)$ can be found

$\Rightarrow u(a, t) = u(b, t)$

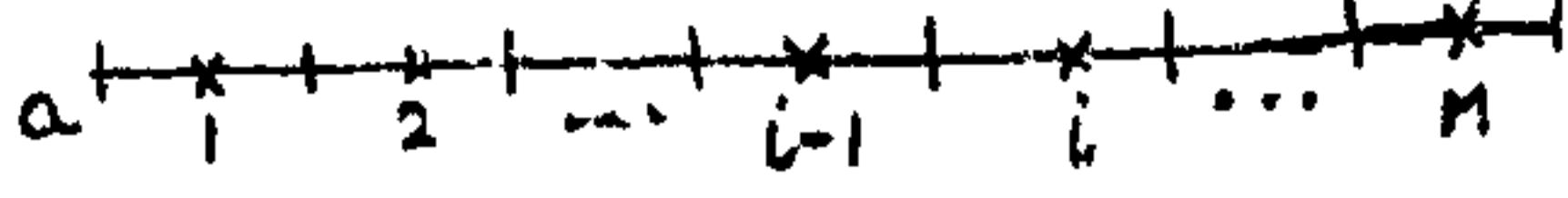
More general "quasilinear" 1st order PDE: $au_x + bu_y = c(x, y, u)$

Characteristics: $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = ds \Rightarrow \left\{ \frac{dx}{ds} = a, \frac{dy}{ds} = b \right. \text{ are characteristic curves (ODE system)}$
determines $C: \begin{cases} x = x(s) \\ y = y(s) \end{cases}$

Then along each characteristic: $\frac{du}{ds}(x(s), y(s)) = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = au_x + bu_y = c$, so $\frac{du}{ds} = c$

Finite Volume Discretization of $\frac{\partial u}{\partial t} + \nabla \cdot \vec{F} = 0$ in 1-D

To make the idea clear, first in 1-D: $\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$ in $\Omega = (a, b)$

mesh: 
 nodes x_i , $i=1, 2, \dots, M$
 faces $x_{i-1/2}$, $i=1, \dots, M+1$
 control volumes $V_i = [x_{i-1/2}, x_{i+1/2}] \times A$

$$\Rightarrow \int_{V_i} u A dx \Big|_{t=t_n}^{t=t_{n+1}} + \int_{t_n}^{t_{n+1}} [AF(x_{i+1/2}, t) - AF(x_{i-1/2}, t)] dt = 0$$

Set $U_i^n := \frac{1}{\Delta V_i} \int_{V_i} u(x, t_n) dx$ = mean value of u over V_i at time t_n

$AF_{i-1/2}^{n+\theta} := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} AF(x_{i-1/2}, t) dt$ = mean flux during time-step across face $i-1/2$

Thus we get exact discrete conservation

$$\Delta V_i [U_i^{n+1} - U_i^n] + \Delta t [AF_{i+1/2}^{n+\theta} - AF_{i-1/2}^{n+\theta}] = 0, \quad i=1, 2, \dots, M \\ n=0, 1, \dots, N$$

Explicit scheme: choose $\theta=0$; assume mean flux during Δt remains

\approx flux at time t_n . Knowing U_i^n , $i=1, \dots, M$ and constitutive law for $F=F(u)$, we can compute $F_{i-1/2}^n$, $i=2, \dots, M$ and the b.c. conditions provide $F_{1/2}^n$, $F_{M+1/2}^n$, so can update

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta V_i} [AF_{i-1/2}^n - AF_{i+1/2}^n], \quad i=1, 2, \dots, M$$

Implicit schemes: choose $0 < \theta \leq 1$ and $F^{n+\theta} = (1-\theta)F^n + \theta F^{n+1}$

Then equations form a coupled system for U_i^{n+1} , will need matrix solvers...

$\theta = \frac{1}{2}$ is the Crank-Nicolson scheme

$\theta = 1$ is the fully implicit (backward Euler) scheme

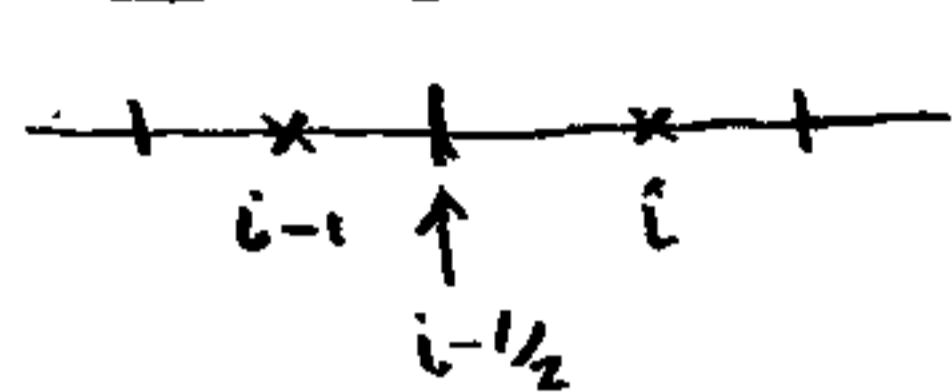
Explicit scheme for pure advection in 1-D: $F = v u$

$v(x,t)$ = velocity (known)

$\Rightarrow F_{i-\frac{1}{2}}^n = (v u)_{i-\frac{1}{2}}^n$, this requires u at face $i-\frac{1}{2}$, whereas the scheme

updates only the mean values U_i , so we need to express face values in terms of mean values.

$$\frac{U_{i-1}}{U_i}$$



$$\text{Simplest choice: } U_{i-\frac{1}{2}} \approx \frac{U_{i-1} + U_i}{2}$$

$$\begin{aligned} U_i^{n+1} &= U_i^n + \frac{v \Delta t}{\Delta x} \left[\frac{U_{i-1} + Y_i}{2} - \frac{U_i + U_{i+1}}{2} \right] \\ &= U_i^n + \frac{v \Delta t}{\Delta x} [U_{i-1} - U_{i+1}] \end{aligned}$$

always has a negative coeff!

bad idea! the resulting scheme is unstable!

and unphysical: if stream flows to the right ($v > 0$) then the face will only see what comes from the left (upstream), not the average of the upstream and downstream values!

The physically correct prescription is the "upwind" flux:

$$F_{i-\frac{1}{2}}^n = \begin{cases} v_{i-\frac{1}{2}}^n U_{i-1}^n & \text{if } v_{i-\frac{1}{2}}^n > 0 \\ v_{i-\frac{1}{2}}^n U_0^n & \text{if } v_{i-\frac{1}{2}}^n < 0 \end{cases} \quad i = 2, \dots, M$$

The resulting explicit upwind scheme will be stable if it preserves positivity ($U_i^n > 0 \Rightarrow U_i^{n+1} > 0$) which requires all coefficients to be positive,

$$\text{e.g. for } v > 0: U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x_i} [v U_{i-1}^n - v U_i^n]$$

$$= \left[1 - \frac{v \Delta t}{\Delta x_i} \right] U_i^n + \frac{v \Delta t}{\Delta x_i} U_{i-1}^n \quad \text{must have } 1 - \frac{v \Delta t}{\Delta x_i} \geq 0$$

$$\Rightarrow \Delta t \leq \frac{\Delta x_i}{v} \quad (\text{CFL condition}) \quad \forall i$$

Similarly for $v < 0$, so the CFL condition is

$$\Delta t \leq \frac{\min \Delta x_i}{\max |v|}$$

Note: in 1-D, $\nabla \cdot \vec{v} = 0 \Rightarrow v \equiv \text{constant physical speed}$

Boundary condition: If $v > 0$ then only the left boundary is an inflow bry, where a bry condition must be prescribed.

Dirichlet BC: $u(a,t) = u_a(t) = \text{given}$. Set $U_0^n = u_a(t_n) = \text{given}, n = 1, 2, \dots$

Then $F_{\frac{1}{2}}^n = v U_0^n$ and the scheme will update $U_i^n, i = 1, \dots, M$

Neumann BC: $F(a,t) = \text{given}$. Then $F_{\frac{1}{2}}^n = F(a, t_n) \Rightarrow U_0^n = \frac{F(a)}{v}$

Similarly, if $v < 0$ then BC only at $x=b$: $F_{M+\frac{1}{2}}^n = v U_b(t_n)$ or $F(b, t_n)$.

$$\Rightarrow U_M^n = \frac{F}{v}$$