## V. Alexiades, PDE LECTURE NOTES, Chapter I

## 4. NONLINEAR ADVECTION

## 4.A. The inviscid Burgers Equation - Shocks

The model (prototype) equation for nonlinear advection is the Inviscid Burgers equation:

$$
\begin{cases}u_{t}+u u_{x}=0, & x \in \mathbb{R}, t>0 \\ u(x, 0)=u_{0}(x), & \text { e.g. } u_{0}(x)= \begin{cases}2, & x<0 \\ 2-x, & 0<x<1 \\ 1 & x>1\end{cases} \end{cases}
$$

Let's see how the inital signal $u_{0}(x)$ propagates.

$$
\text { Here } \Gamma_{0}:\left\{\begin{array}{l}
x=x_{0}(s)=s \\
t=y_{0}(s)=0 \\
u=u_{0}(s)
\end{array} \quad, \quad \frac{d X}{d \tau}=U, \frac{d Y}{d \tau}=1, \frac{d U}{d \tau}=0\right.
$$

$\Rightarrow U(\tau, s)=$ const. $=U(0, s)=u_{0}(s), t=Y(\tau, s)=\tau+c_{1} \Rightarrow t=Y(\tau, s)=\tau, x=x(\tau, s)=$ $u_{0}(s) \tau+c_{2} \Rightarrow x=X(\tau, s)=u_{0}(s) \tau+s$
so $x=X(\tau, s)=u_{0}(s) \cdot \tau+s, t=Y(\tau, s)=\tau, u=U(\tau, s)=u_{0}(s) \Rightarrow x=u_{0}(s) t+s$ are straight lines emanating from $(x=s, t=0)$ and $u$ is given implicitly by $u=u_{0}(x-u t)$.

If $u_{0}(s) \equiv$ const. then the characteristics are parallel lines, otherwise they may intersect at $(x, t)$ such that: $u_{0}\left(s_{1}\right) t+s_{1}=x=u_{0}\left(s_{2}\right) t+s_{2} \Rightarrow$ at $t=\frac{s_{2}-s_{1}}{u_{0}\left(s_{1}\right)-u_{0}\left(s_{2}\right)}$. If this $t$ value is $>0$ then the characteristics intersect at some $t>0$, in which case $u$ will become discontinuous, hence no classical solution! A shock forms. Note that this will happen if $u_{0}(x)$ is nonincreasing, as in our example above. Indeed, the speed being $u$, large values of $u$ propagate faster than small values, so an initial nonincreasing profile $\left(u_{0}^{\prime}(x)<0\right)$ will steepen till a shock forms; whereas an increasing initial profile will flatten to a rarefaction wave (when moving to the right).

Let's find the time at which a shock forms (when $u_{0}^{\prime} \leq 0$ ) by looking at $u_{x}$ to see when a singularity can form: compute $u_{x}$ along a characteristic $x-u t=s: u=u_{0}(x-u t) \Rightarrow u_{x}=u_{0}^{\prime}(s) \cdot\left[1-u_{x} \cdot t\right] \Rightarrow u_{x}=$ $\frac{u_{0}^{\prime}(s)}{1+u_{0}^{\prime}(s) \cdot t}$ which becomes infinite when $t=-\frac{1}{u_{0}^{\prime}(s)}$, which is $>0$ only when $u_{0}^{\prime}(s)<0$ as we already know. The first positive time at which this happens corresponds to a minimum of $u_{0}^{\prime}(s)$. At that moment $u_{x}$ blows up and no $\mathcal{C}^{1}$ solution exists after that time!

## Example:

$$
u_{0}(x)=\left\{\begin{array}{ll}
2 & , x \leq 0 \\
2-x & , 0<x<1 \\
1 & , x \geq 1
\end{array} \quad \Rightarrow \quad u_{0}^{\prime}(s)=\left\{\begin{array}{l}
0 \\
-1 \text { is }<0 \\
0
\end{array}\right.\right.
$$

someplace so a shock will form, at time $t^{*}=-\frac{1}{-1}=1$.
Plot the characteristics:

$$
x=u t+s, \quad u(x, t)=u(s, 0)=u_{0}(s)=\left\{\begin{array}{l}
2 \quad, s \leq 0 \\
2-s \quad, 0<s<1 \\
1,1 \leq s
\end{array}\right.
$$

For $s<0: u=2$ so speed $2=\frac{d x}{d t}$; for $s>1: u=1$; for $0<s<1: u=2-s$
Solution up to $t=1$ : For $x<2 t: u=2, \quad$ for $x>t+1: u=1$,

$$
\text { for } \begin{aligned}
2 t<x<t+1: \quad x & =(2-s) t+s \Rightarrow s=\frac{x-2 t}{1-t} \\
& u=u_{0}(s)=2-s=2-\frac{x-2 t}{1-t}=\frac{2-x}{1-t}
\end{aligned}
$$

At $t \geq 1$, the characteristics intersect, $u$ becomes discontinuous, so no $\mathcal{C}^{1}$ (classical) solutions exist.
Important observation: "Hyperbolic equations are non-smoothing," i.e. non-smooth features (discontinuities in $u, u_{x}$, etc.) are propagated into the future (no smoothing occurs). Linear equations propagate discontinuities along characteristics, nonlinear ones may propagate them along non-characteristics too (see shock curve, nextpage). This is in great contrast to parabolic PDEs which are "smoothing."

## 4.B. Weak solution formulation

What happens after $t=1$ ? Does the world come to an end? Well, only the world of classical solutions! Physically, the wave "breaks", but surely some sort of solution must exist! We are led to look for solutions which are not necessarily $\mathcal{C}^{1}$. Since it is the classical pointwise interpretation of the PDE that breaks down, we revert back to the general integral form of the conservationa law, which makes sense even for discontinuous $u$. Thus, we follow Rule No. 1 in PDE: If you don't know what to do next, integrate by parts! i.e. transfer the derivatives away from $u$.

First, we rewrite the PDE in "conservation form": $u_{t}+\left(\frac{1}{2} u^{2}\right)_{x}=0$. More generally, let's consider the general quasilinear conservation law

$$
\begin{equation*}
u_{t}+\frac{\partial}{\partial x} F(u)=0, \quad u(x, 0)=u_{0}(x) \tag{1}
\end{equation*}
$$

Multiply by a smooth (test) function $\varphi(x, t)$, which we'll choose later,

$$
u_{t} \varphi+(F(u))_{x} \varphi=0 \Rightarrow(u \varphi)_{t}-u \varphi_{t}+[F(u) \varphi]_{x}-F(u) \varphi_{x}=0
$$

and integrate over $Q=(a, b) \times(0, \hat{t})$

$$
\begin{aligned}
& \int_{0}^{\hat{t}} \int_{a}^{b}\left\{(u \varphi)_{t}+[F(u) \varphi]_{x}\right\} d x d t-\iint_{Q}\left\{u \varphi_{t}+F(u) \varphi_{x}\right\} d x d t=0 \\
\Rightarrow & \int_{\partial Q} u \varphi n_{t} d s+\int_{\partial Q} F(u) \varphi n_{x} d s-\iint_{Q}\left\{u \varphi_{t}+F(u) \varphi_{x}\right\} d x d t=0 \\
\Rightarrow & \iint_{Q}\left\{u \varphi_{t}+F(u) \varphi_{x}\right\} d x d t=\left.\int_{a}^{b} u \varphi\right|_{t=0}(-1) d x+\left.\int_{a}^{b} u \varphi\right|_{t=\hat{t}}(1) d x+\left.\int_{0}^{\hat{t}} F(u) \varphi\right|_{x=a} ^{x=b} d t
\end{aligned}
$$

Now $u(x, 0)=u_{0}(x)$, but $\left.u\right|_{t=\hat{t}},\left.u\right|_{x=a, b}$ are not known from the CP so we choose the test functions $\varphi$ to vanish there:

Lemma. $A$ smooth solution $u$ of the $C P: u_{t}+F(u)_{x}=0, u(x, 0)=u_{0}(x),-\infty<x<\infty, t>0$ satisfies

$$
\begin{equation*}
\int_{0}^{\hat{t}} \int_{a}^{b}\left\{u \varphi_{t}+F(u) \varphi_{x}\right\} d x d t+\int_{a}^{b} u_{0}(x) \varphi(x, 0) d x=0 \tag{2}
\end{equation*}
$$

for any interval $(a, b), \hat{t}>0$ and $\varphi \in \mathcal{C}^{1}(\bar{Q})$ such that $\left.\varphi\right|_{t=\hat{t}}=\left.\varphi\right|_{x=a}=\left.\varphi\right|_{x=b} \equiv 0$.
Note that this integral relation makes sense for any $u$ for which the integral exists.
Definition: We say $u$ is an $L^{2}$ weak solution of the $C P$ in $\mathbf{Q}=(a, b) \times(0, \hat{t})$ if $u, F(u) \in L^{2}(Q)$ and

$$
\begin{equation*}
\iint_{Q}\left\{u \varphi_{t}+F(u) \varphi_{x}\right\} d x d t+\int_{a}^{b} u_{0}(x) \varphi(x, 0) d x=0 \tag{3}
\end{equation*}
$$

for any $\varphi \in \Phi:=\left\{\varphi \subset \mathcal{C}^{1}(\bar{Q}):\left.\quad \varphi\right|_{x=a}=\left.\varphi\right|_{x=b}=\left.\varphi\right|_{t=\hat{t}}=0\right\} \supset \mathcal{C}_{0}^{\infty}(Q)$.
The previous Lemma says that a classical solution is also a weak solution. Conversely, it is easy to see that a smooth weak solution is classical, i.e. satisfies (1).

Problem: Show that if $u$ is an $L^{2}$-weak solution of the CP in $Q=(a, b) \times(0, \hat{t})$ and if $u \in \mathcal{C}^{1}(Q) \cap \mathcal{C}(\bar{Q})$ then it is also a classical solution of the CP, i.e. satisfies (1). [Choose $\varphi \in \Phi, \varphi(x, 0)=0$, integrate by parts to show PDE holds; then show the initial condition holds].

The danger in enlarging the concept of solution is that if we overdo it we may loose uniqueness. In fact, this is the case here, as we shall see, and we need to impose an "entropy inequality" in order to guarantee uniqueness!

Now that we have a concept of solution that would make sense even after a shock forms, let's try to find it beyond $t^{*}=1$ in our example. We do know the solution even for $t>t^{*}=1$ except in the region between the $s=0$ and $s=1$ characteristics. Physically, we expect a shock to form, a single curve $x=\Sigma(t)$ across which $u$ jumps from $u=2$ on its left to $u=1$ on its right. Can we find such a curve ?

Let's examine what condition a shock must satisfy: We consider a weak solution $u$ which is smooth in $Q$ except possibly across a $\mathcal{C}^{1}$ curve $x=\Sigma(t)$ where it has a jump. For any $\varphi \in \mathcal{C}_{0}^{\infty}(Q) \subset \Phi$ we have from (3):

$$
\begin{aligned}
0= & \iint_{Q}\left\{u \varphi_{t}+F(u) \varphi_{x}\right\} d x d t=\iint_{Q^{-}}\{\quad\}+\iint_{Q^{+}}\{\quad\} \text { in each } Q_{i} u \text { is smooth so } \\
= & \left(\int_{\partial Q^{-}}+\int_{\partial Q^{+}}\right)\left(u \varphi n_{t}+F(u) \varphi n_{x}\right) d s-\left(\iint_{Q^{-}}+\iint_{Q^{+}}\right)\left(u_{t}+\left(F(u)_{x}\right) \varphi d x d t\right. \\
= & \int_{x=\Sigma(t)}\left(u^{-} \varphi(-d x)+F\left(u^{-}\right) \varphi d t\right)-\int_{x=\Sigma(t)}\left(u^{+} \varphi(-d x)+F\left(u^{+}\right) \varphi(+d t)\right)+0 \\
& \quad\left(\varphi=0 \text { on the rest of } \partial Q^{-}, \partial Q^{+}\right) \\
0= & \int_{x=\Sigma(t)} \varphi\left(\left[u^{+}-u^{-}\right] d x-\left[F\left(u^{+}\right)-F\left(u^{-}\right)\right] d t\right)=\int_{x=\Sigma(t)} \varphi\left([u]_{-}^{+} \frac{d x}{d t}-[F(u)]_{-}^{+}\right) d t
\end{aligned}
$$

$\forall \varphi \in \mathcal{C}_{0}^{\infty}(Q) \Rightarrow$ the integrand must vanish and we obtain the

$$
\text { Rankine-Hugoniot or shock condition: } \quad[u]_{-}^{+} \Sigma^{\prime}(t)=[F(u)]_{-}^{+}
$$

which in fact determines the speed of the shock. In general, the shock curve $x=\Sigma(t)$ is not a characteristic, contrary to the case of linear equation where discontinuities can only propagate along characteristics.

Classical formulation of shock problem when there is a smooth shock curve: We can separate the problem into two, one on each side of the shock: Find $u^{+}, u^{-}, \Sigma(t): u_{t}^{-}+F^{-}\left(u^{-}\right)_{x}=0$ in $x<$ $\Sigma(t), u_{t}^{+}+F^{+}\left(u^{+}\right)_{x}=0$ in $x>\Sigma(t),[u]_{-}^{+} \Sigma^{\prime}(t)=[F(u)]$. This is the traditional approach in phase transition problems. Note that, in this approach, you have to know the structure of the answer in order to formulate the question!

## 4.C. Construction of the shock solution

Example continued: Let's apply the shock condition to our example to find the solution for time $t>t^{*}=1$. We have $[u]_{-}^{+}=u^{+}-u^{-}=1-2=-1,[F(u)]_{-}^{+}=\left[\frac{u^{2}}{2}\right]_{-}^{+}=\frac{1^{2}}{2}-\frac{2^{2}}{2}=-\frac{3}{2}$, so $(-1) \Sigma^{\prime}(t)=-\frac{3}{2} \Rightarrow \Sigma^{\prime}(t)=\frac{3}{2}$ (more generally, $\left(u^{+}-u^{-}\right) \Sigma^{\prime}=\frac{\left(u^{+}\right)^{2}-\left(u^{-}\right)^{2}}{2} \Rightarrow \Sigma^{\prime}(t)=\frac{u^{+}+u^{-}}{2}$ ) so the shock is a line with speed $\frac{3}{2}$ through $(x=2, t=1)$, i.e. $t-1=\frac{2}{3}(\Sigma-2)$ i.e. $x=\Sigma(t)=\frac{1+3 t}{2}$. On its left $u \equiv u^{-}=2$, on its right $u=u^{+}=1$.

This is the unique (weak) solution of the example problem, which exists for all time.

## 4.D. Example of a rarefaction wave

Consider

$$
\begin{aligned}
& u_{t}+u u_{x}=0 \\
& u(x, 0)=u_{0}(x)=\left\{\begin{array}{l}
u^{-}, x<0 \\
u^{+}, x>0
\end{array}, u^{ \pm}=\text {const. }, u^{-}<u^{+} .\right.
\end{aligned}
$$

Now the characteristics open up instead of intersecting, is $u$ discontinuous for any $t>0$ ?
We can construct infinitely many weak solutions here!
e.g. with one shock : $\Sigma^{\prime}=\frac{u^{-}+u^{+}}{2}, u=\left\{\begin{array}{l}u^{-} \text {for } x<\frac{u^{-}+u^{+}}{2} t \\ u^{+} \text {for } x>\frac{u^{-}+u^{+}}{2} t\end{array}\right.$
e.g. with two schocks : $\Sigma_{1}^{\prime}=\frac{u^{-}+u^{*}}{2}, \Sigma_{2}^{\prime}=\frac{u^{*}+u^{+}}{2}$ so $u=\left\{\begin{array}{l}u^{-}, x<\frac{u^{-}+u^{*}}{2} t \\ u^{*}, \frac{u^{-}+u^{*}}{2} t<x<\frac{u^{*}+u^{+}}{2} t \\ u^{+}, x>\frac{u^{*}+u^{+}}{2} t\end{array}\right.$
for any $u^{-}<u^{*}<u^{+}$
e.g. we can even fit $N$ shocks the same way
and there is a solution without shocks, a fan solution, called rarefaction wave

$$
\begin{aligned}
u & = \begin{cases}u^{-}, & x<u^{-} t \\
\frac{x}{t}, & u^{-} t<x<u^{+} t \\
u^{+}, & x>u^{+} t\end{cases} \\
u-u^{-} & =\frac{u^{+}-u^{-}}{u^{+} t-u^{-} t}\left(x-u^{-} t\right)=\frac{x}{t}-u^{-}
\end{aligned}
$$

Which one is physically correct??? Only the rarefaction solution seems reasonable, but what's wrong with the shock solutions? They are in fact weak solutions! It turned out that in the gas dynamics case, the entropy would decrease across such a shock solution, violating the 2nd Law of Thermodynamics. A condition that allows only the rarefaction wave is the "entropy inequality": $F^{\prime}\left(u^{+}\right) \leq \Sigma^{\prime} \leq F^{\prime}\left(u^{-}\right)$ which for Burgers is $u^{+} \leq \Sigma^{\prime} \leq u^{-}$, which excludes shocks when $u^{-}<u^{+}$[Peter Lax, Conservation Laws and Shock Waves, SIAM, 1973 ].

Physically acceptable solutions are also obtained as limits of "viscosity solutions" of the viscous Burgers' $u_{t}+u u_{x}=\varepsilon u_{x x}$ as $\varepsilon \rightarrow 0$. Their theory has been generalized dramatically in the 1980s.

