## 8. STABILITY IDEAS

## 8.A. Introduction

Definition: We say an Initial Value Problem is stable if any solution that is initially bounded remains bounded as $t \rightarrow \infty$. If there exists solutions which are initially bounded but can become unbounded as $t \rightarrow \infty$, then we say the (IVP) is unstable.

These concepts are particulary crucial in numerical methods. Unstable methods are useless because errors (perturbations), being unavoidable, will grow and render the answers useless.

Instability indicates that there are internal mechanisms in the equation which can make pertubations grow as $t \rightarrow \infty$. It can often be detected by normal mode analysis, by seeing what happens to a wave.

## 8.B. Harmonic waves

Periodic waves are of great interest and usefulness. They are usually considered in the form:

$$
\begin{aligned}
u(x, t) & =a e^{i[k x-\omega t]}, \quad a, k \text { real } \\
& =a[\cos (k x-\omega t)+i \sin (k x-\omega t)]
\end{aligned}
$$

Meaning of terms and terminology associated with harmonic waves:
$a=$ amplitude
$k=$ wavenumber $=$ number of oscillations in $2 \pi$ units of space at fixed time
$\omega=$ angular frequency $=$ number of oscillations in $2 \pi$ units of time at fixed $x$
$\lambda=\frac{2 \pi}{k}=$ wavelength, $\frac{1}{\lambda}=\frac{k}{2 \pi}=$ number of oscillations per unit length of space
$p=\frac{2 \pi}{\omega}=\operatorname{period}$ (in time), $f=\frac{1}{p}=\frac{\omega}{2 \pi}=$ frequency $=$ number of oscillations per unit time
$c=\frac{\omega}{k}=\lambda f=$ phase velocity $=$ velocity of a fixed point on the wave.
$u=A e^{i k\left(x-\frac{\omega}{k} t\right)}=$ wave traveling to the right with speed $c=\frac{\omega}{k}$

## 8.C. Normal mode analysis for linear 2nd order PDEs

To gain insight into the behavior of various terms in evolution equations, and into the behavior of the equations themselves, it is useful to see what happens to a "wave" input (initial condition), by seeking solutions of the form

$$
\begin{equation*}
\text { Normal mode: } \quad u_{k}(x, t)=a(k) \exp \{i(k x-\omega t)\} \tag{1}
\end{equation*}
$$

$a=$ amplitude, $k=$ wave number $\in \mathbb{R}, \omega=$ angular frequency, $\lambda=\frac{2 \pi}{k}=$ wavelength and find $\omega(k)$ so as to satisfy the equation.

Consider the linear 2nd order homogeneous equation (no external forcing)

$$
\begin{equation*}
A u_{x x}+2 B u_{x t}+C u_{t t}+D u_{x}+E u_{t}+F u=0 \tag{2}
\end{equation*}
$$

with constant coefficients. The PDE is classified as

$$
\begin{array}{rll}
>0 & \text { hyperbolic } \\
B^{2}-A C & =0 & \text { parabolic } \\
<0 & \text { elliptic }
\end{array}
$$

Plug $u=u_{k}$ into (2). To be a solution, we find that $\omega$ must satisfy the quadratic

$$
\begin{equation*}
\text { dispersion relation: } C \omega^{2}-[2 B k-E i] \omega+\left[A k^{2}-D i k-F\right]=0 \tag{3}
\end{equation*}
$$

For such $\omega(k)=\omega_{k}$, the normal modes are

$$
\begin{equation*}
u_{k}(x, t)=a(k) e^{i\left[k x-\operatorname{Re} \omega_{k} \cdot t\right]} e^{\left(\operatorname{Im} \omega_{k}\right) t} \tag{4}
\end{equation*}
$$

with magnitude $\left|u_{k}\right|=|a(k)| e^{\left(\operatorname{Im} \omega_{k}\right) t}$. Assuming $a(k)$ is bounded $\forall k \in \mathbb{R}$, the growth or decay of $\left|u_{k}\right|$ in time is determined by $\operatorname{Im} \omega(k),-\infty<k<\infty$, i.e. by the number

$$
\Omega:=\sup _{-\infty<k<\infty} \operatorname{Im} \omega(k)
$$

Either $\Omega=+\infty(\omega(k)$ unbounded), or $\Omega<\infty(\omega(k)$ bounded). We examine the (IVP) for (2) in each case.

Case I: $\Omega=+\infty$, i.e. $\operatorname{Im} \omega(k)$ unbounded. Then $\frac{1}{\left|\omega_{k}\right|}$ is bounded and can be made arbitrarily small (for $k$ large enough). Choosing $a(k)=\frac{1}{\left|\omega_{k}\right|^{2}}$ (bounded), the normal modes initially are

$$
u_{k}(x, 0)=\frac{1}{\left(\omega_{k}\right)^{2}} e^{i k x}, \quad \frac{\partial}{\partial t} u_{k}(x, 0)=\frac{i}{\omega_{k}} e^{i k x}
$$

which can be made arbitrarily small in magnitude (for large $k$ ), whereas for any $\mathbf{t}>\mathbf{0}$

$$
\left|u_{k}(x, t)\right|=\left|\frac{1}{\omega_{k}^{2}}\right| e^{\left(\operatorname{Im} \omega_{k}\right) t}
$$

can be made arbitrarily large (for large $k$ ). Hence, a small perturbation of the $u \equiv 0$ solution can grow large even in finite time. We see that such a Cauchy problem violates continuous dependence on (initial) data, so the problem is not well-posed. And since initially bounded solutions can grow unbounded as $t \rightarrow \infty$, the problem is also unstable.

Case II: $\Omega:=\sup _{k} \operatorname{Im} \omega(k)<\infty$. Then $\left|u_{k}(x, t)\right|$ cannot become unbounded in finite $t$ and the Cauchy problem will be well-posed (as it can be shown). Stability depends on the sign of $\Omega$.
(i) If $\Omega<0$ then (2) is strictly stable: $\left|u_{k}(x, t)\right| \rightarrow 0$ as $t \rightarrow \infty \forall k \in \mathbb{R}$.
(ii) If $\Omega=0$ then (2) is neutrally stable: there may exist modes (values of $k$ ) for which $\left|u_{k}\right|$ is bounded but $\nrightarrow 0$ as $t \rightarrow \infty$, while all other $\left|u_{k}\right| \rightarrow 0$ as $t \rightarrow \infty$. If there is instability, the growth will be algebraic, not exponential.
(iii) If $\Omega>0$ then (2) is unstable: there are modes ( $k$ near $\Omega$ ) for which $\left|u_{k}\right| \rightarrow \infty$ as $t \rightarrow \infty$.

In the neutrally stable case, $\Omega=0$, if $\operatorname{Im} \omega(k)=0 \forall k \in \mathbb{R}$ i.e. if all $\omega(k)$ are real, then there is no decay,

$$
\left|u_{k}(x, t)\right|=\left|u_{k}(x, 0)\right| \quad \forall k
$$

so also the energy $\int|u|^{2} d x$ is conserved and the PDE is said to be of conservative type. Furthermore, if $\omega^{\prime \prime}(k) \neq 0(\omega(k)$ is not a linear function of $k)$ then the PDE is called dispersive, else nondispersive, (see example below), because the phase speed $c=\frac{d x}{d t}=\frac{\omega(k)}{k} \neq$ constant, so different modes propagate with different speeds.

When $\Omega \leq 0$ and $\operatorname{Im} \omega(k)<0$ for all except possibly for a finite number of $k$ 's, then $\left|u_{k}(x, t)\right| \searrow 0$ as $t \rightarrow \infty$ for all but finitely many $k$ 's, so the energy $\int|u|^{2} d x$ dissipates and the PDE is dissipative.

The group velocity of a wave packet is: $v_{g}=\frac{d \omega(k)}{d k}=\frac{d}{d k}(k c)=c+k \frac{d c}{d k}$ where $c$ is the phase speed of constituent waves. The medium is non-dispersive if $v_{g}=c$ i.e. if $\frac{d c}{d k}=0$ in which case the pulse maintains its shape as it travels.

## 8.D. Examples

(i) Elliptic equations: $u_{x x}+u_{t t}+q u=0, q=$ const. Here the dispersion relation is $\omega^{2}+k^{2}-q=0 \Rightarrow$ $\omega= \pm \sqrt{q-k^{2}}$, so for $k^{2} \geq q$ we have $\operatorname{Im} \omega= \pm \sqrt{k^{2}-q} \Rightarrow \Omega=+\infty$. Hence the Cauchy problem is not well-posed (Hadamard's example shows it explicitly).
(ii) Parabolic equations: $u_{t}+c u_{x}=\alpha u_{x x}, c, \alpha$ constants. Here $\omega=-\alpha k^{2} i+c k \Rightarrow \operatorname{Im} \omega=-\alpha k^{2}$, so if $\alpha>\mathbf{0}$ (heat equation, advection-diffusion equation) then $\Omega=0$, the Cauchy problem is well-posed, and neutrally stable. Normal modes are

$$
u_{k}(x, t)=a(k) e^{i k(x-c t)} e^{-\alpha k^{2} t}
$$

consisting of a harmonic wave of speed $c$ (driven by the advective term $c u_{x}$ ) and an exponentially decaying amplitude (due to the diffusion term $\alpha u_{x x}$ ). The larger the diffusivity $\alpha>0$, the faster each mode decays, and modes of small wavelength $\lambda=\frac{2 \pi}{k}$ decay faster than modes of large wavelength. Since all modes $\rightarrow 0$ as $t \rightarrow \infty$, the equation is strictly stable.
If $\alpha<0$ (backward heat equation) then $\Omega=+\infty$, so the Cauchy problem is not well-posed (take $u_{k}(x, 0)=\frac{1}{-\alpha k^{2}} e^{i k x}=$ small, but $\left|u_{k}(x, t)\right|=\frac{1}{-\alpha k^{2}} e^{-\alpha k^{2} t}=$ large) so backward diffusion $(t \rightarrow-t$ amounts to $\alpha \rightarrow-\alpha$ ) is ill-posed as is clear physically: a drop of ink in a glass of water will diffuse everywhere, it is impossible to retrace back to initial location of the drop.
Important remark: parabolic PDEs distinguish strictly between past and future (unlike hyperbolic which are invariant). Only the future can be determined not the past! This is a crucial and distinguishing property of parabolic problems.
(iii) Hyperbolic equations: $u_{t t}-c^{2} u_{x x}+q u=0, q=$ const., $c=$ const. Now $\omega= \pm \sqrt{c^{2} k^{2}+q}$, so if $q \geq 0$ then $\operatorname{Im} \omega_{k}=0 \forall k$ (no decay, conservative type PDE since $\left|u_{k}(x, t)\right|=\left|u_{k}(x, 0)\right|$, so energy is constant) and $\Omega=0$ (neutrally stable). If $q<0$ then $\Omega<\infty$, the Cauchy problem is well-posed always. The Klein-Gordon equation $\left(q=m^{2} c^{4} / h^{2}>0\right)$ and the wave equation $(q=0)$ are neutrally stable and conservative, with normal modes the left and right traveling waves

$$
u_{k}(x, t)=a(k) e^{i k\left(x \pm \sqrt{\left.c^{2}+\frac{q}{k^{2}} t\right)}\right.}
$$

The Klein-Gordon equation is dispersive since $w^{\prime \prime}(k) \neq 0$, but the wave equation is nondispersive since $\omega(k)= \pm c k$ is linear in $k$. The damped wave equation $u_{t t}-c^{2} u_{x x}+q u_{t}=0$ is dissipative.

Remark: For IBVP's, the boundary conditions may restrict $k$ to a discrete set (spectrum). Then the solution is a superposition of these modes. If one or more of these modes is unbounded as $t \rightarrow \infty$ the problem is unstable. If they all decay it is stable.

Problem 4: Carry out a normal mode analysis for each of the following:
(i) $u_{t t}+c u_{x x x x}=0, c \gtrless 0$.
(ii) $u_{t}+c u_{x}+q u_{x x x x}=0, q>0$.

Problem 5: Discuss normal modes and stability for the telegrapher's equation

$$
u_{t t}-c^{2} u_{x x}+2 q u_{t}=0, \quad q \gtrless 0
$$

Justify why it is called the "damped wave equation".

