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CHAPTER I

FIRST ORDER PDEs

1. INTRODUCTION

1.A. 1st order PDE

The general 1st order PDE for the unknown $u(x, y, \dots, z)$ has the form

$$F(x, y, \dots, z,\ u_x, u_y, \dots, u_z) = 0.$$

Such equations arise in Hamilton-Jacobi theory (calculus of variations), particle mechanics, geometrical optics. We saw that conservation laws produce PDEs of the form (actually systems usually)

$$u_t + \frac{\partial}{\partial x} a(x,\, t,\, u) = b(x,\, t,\, u)$$

which are generically referred to as "conservation law form" (even if a is not the flux of u).

The theory of 1st order PDEs is essentially complete (contrary to the situation for any other general class of PDEs), due to the fact that they can be reduced to solving systems of ODEs. It is a **local theory** (just like ODE theory) and very geometrical, unlike the rest of PDE theory.

We may think of the solution $u(x, y, \dots, z)$ as a (hyper)surface, to be constructed, called an **integral surface**. We expect many solutions, a family of surfaces depending on an arbitrary function, not just on an arbitrary constant as for ODEs. How do we select an individual integral surface out of such a large family of surfaces? For an ODE, the general solution is a family of curves and we select one by requiring that it passes through a given "initial" point. Here the general solution is a family of surfaces, so naturally we can pick one by making it pass through a given "initial" curve Γ_0.

The Cauchy Problem for a 1st order PDE is the problem of finding a solution (integral) surface of the PDE which also passes through a given "initial" curve Γ_0.

1.B. Wave propagation

A **wave** is a disturbance propagating in time through a medium, carrying energy, e.g. electromagnetic waves, sound waves, water waves, seismic waves. Matter is not necessarily convected with the wave, it is the disturbance carrying energy that propagates.

Mathematical model of a wave: $u(x, t) = F(x - ct)$

undistorted wave traveling to the right with speed $c > 0$, (or to the left if $c < 0$).

What equation does $u(x, t)$ satisfy?

$$u_t + F' \cdot (-c)\ ,\ u_x = F' \quad \Rightarrow \quad u_t + cu_x = 0 \quad \text{1st order linear PDE}$$

The **linear advection equation** $u_t + cu_x = 0$ has general solution $u(x, t) = F(x - ct)$, with $F(\cdot)$ an arbitrary functions of one variable.

If we knew the initial shape of the wave: $u(x, 0) = f_0(x)\ ,\ -\infty < x < \infty$ then $F(x) = f_0(x) \ \ \forall x$ and therefore the solution of the **Initial Value Problem**, or **Cauchy Problem**

$$(CP) \begin{cases} u_t + cu_x = 0\ ,\ -\infty < x < \infty,\ t > 0 \\ u(x, 0) = f_0(x)\ ,\ -\infty < x < \infty \end{cases}$$

would be $u(x, t) = f_0(x - ct)$, provided $f_0 \subset \mathcal{C}^1(\mathbb{R})$.

Indeed, along $x - ct =$ const. we have $\frac{du}{dt}(x(t), t) = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} = cu_x + u_t = 0$ by the PDE.

The lines $x - ct =$ const. are called the **characteristic curves** or simply **characteristics** of the PDE $u_t + cu_x = 0$.

Note that the direction of the characteristics is $(dx, dt) = (c, 1)$ and

$$u_t + cu_x = (c, 1) \cdot (u_x, u_t) = (c, 1) \cdot \text{grad } u$$

is the **directional derivative** in this direction. So the PDE says that u does **not** change in the direction of the characteristics! This tells us how to solve the PDE:

Given $u_t + cu_x = 0$ or $(c, 1) \cdot (u_x, u_t) = 0 \Rightarrow$ characteristic direction is $(dx, dt) = (c, 1) \Rightarrow \frac{dx}{dt} = \frac{c}{1} = c \Rightarrow x = ct + \xi$, ξ = arbitrary constant, and then along such a characteristic curve: $\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = 0$ is an ODE $\Rightarrow u = k(\xi) \Rightarrow u(x,t) = k(x - ct)$, $k(\cdot)$ arbitrary. This is the **method of characteristics** by which the PDE is reduced to an ODE along each characteristic. That's how we solve 1st order PDEs.

Sinusoidal or Fourier waves

Periodic waves are of great interest and usefulness:

$$u(x,t) = A \sin(kx - wt) \quad \text{or more conveniently} \quad u(x,t) = Ae^{i(kx - wt)}$$

A = amplitude
k = wave number = number of oscillations in 2π units of space at fixed time.
w = angular frequency = number of oscillations in 2π units of time at fixed x
$p = \frac{2\pi}{w}$ = period(in time), $\frac{w}{2\pi} = \frac{1}{P} = \nu$ = frequency = number of oscillations per unit time.
$\lambda = \frac{2\pi}{k}$ = wavelength, $\frac{k}{2\pi}$ = number of oscillations per unit length

Note that $u = Ae^{ik(x - \frac{w}{k}t)}$ is a wave traveling to the right with speed $c = \frac{w}{k}$ = phase velocity = velocity of a fixed point on the wave.

2. THE CAUCHY PROBLEM FOR SEMILINEAR 1ST ORDER PDEs IN 2 VARIABLES

2.A. The Problem

We seek a solution (integral) surface $\Sigma : z = u(x,y)$ satisfying

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{1}$$

and passing through a **given curve** Γ_0, which may be represented parametrically as

$$\Gamma_0 : \quad x = x_0(s),\ y = y_0(s),\ z = u_0(s),\ s \in I = \text{interval of } \mathbb{R} \tag{2}$$

with $(x_0(s),\ y_0(s)) \in \Omega,\ s \in I$, so we want $u(x,y)$ to satisfy

$$u(x_0(s),\ y_0(s)) = z_0(s),\ s \in I. \tag{3}$$

(1),(2),(3) constitute the Cauchy Problem for the PDE (1).

At this point we don't know how smooth the data $a(x,y),\ b(x,y),\ c(x,y,z),\ x_0(s),\ y_0(s),\ z_0(s)$ should be, so we assume them smooth enough for whatever we do to be valid! Since however u is to be $\mathcal{C}^1$ in Ω, the curve Γ_0 should be $\mathcal{C}^1$, so we assume $x_0, y_0, z_0 \in \mathcal{C}^1(I)$.

Note that (1) says $(a,\ b,\ c) \cdot (u_x,\ u_y,\ -1) = 0$, and the vector $(u_x,\ u_y,\ -1)$ is **normal** to the surface $\Sigma : z = u(x,y)$, so (1) says Σ must be, at each point, **tangent** to the "characteristic direction" (a, b, c). We try to contruct the integral surface Σ as union of curves having the characteristic direction (a, b, c) at each point, starting from the given initial curve Γ_0 which is to lie on Σ.

2.B. Method of characteristics

Note that $au_x + bu_y = (a, b) \cdot \nabla u =$ directional derivative of $u(x, y)$ in the direction of the vector (a, b). Since the tangent vector to a curve on the xy plane is (dx, dy), a curve with direction $(dx, dy) = (a, b)$ will have slope $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$, or $\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}$.

Definition: A curve Γ' (on the xy-plane) is called a **characteristic** of the PDE $au_x + bu_y = c$ if it has direction (a, b), i.e. if

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}, \qquad \text{or } \frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}. \tag{4}$$

Written parametrically, in terms of some arbitrary parameter τ, a characteristic

$$\Gamma' : \begin{cases} x = X(\tau) \\ y = Y(\tau) \end{cases}$$

of $au_x + bu_y = c$ satisfies

$$\begin{cases} \frac{dX}{d\tau} = a(X, Y) \\ \frac{dY}{d\tau} = b(X, Y) \end{cases} \tag{4'}$$

Along a characteristic Γ', the PDE gives

$$\frac{du}{d\tau}(X(\tau), Y(\tau)) = \frac{\partial u}{\partial x}\frac{dX}{d\tau} + \frac{\partial u}{\partial y}\frac{\partial Y}{\partial \tau} = au_x + bu_y = c(X, Y, u(X, Y)),$$

so the PDE reduces to an ODE: $\frac{du}{d\tau} = c(x, y, u(x, y))$.

Now let's look at it backwards. If the system of ODEs

$$\begin{cases} \frac{dx}{d\tau} = a(x, y) \\ \frac{dy}{d\tau} = b(x, y) \\ \frac{du}{d\tau} = c(x, y, u) \end{cases} \tag{5}$$

can be solved for $x, \ y, \ u$, we find a curve

$$x = X(\tau), \quad y = Y(\tau), \quad u = U(\tau)$$

in xyu-space, which by construction will have the characteristic direction (a, b, c).

Definition: A curve Γ in xyu-space with characteristic direction, i.e. a solution of (4), or equivalently, of

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)} (= d\tau) \tag{6}$$

is called an **integral curve** of the PDE $au_x + bu_y = c$.

Clearly, a surface that is a union of integral curves will be an integral surface because at each point it will be tangent to the tangent of an integral curve which has characteristic direction!

Example: Solve the (linear) PDE: $xu_x + yu_y = \alpha u, \ \alpha \neq 0$ const.
The ODE system is $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$. The characteristics are: $\frac{dx}{x} = \frac{dy}{y} \Rightarrow ln\, x = ln\, y - ln\, k, \ k =$ arbitrary constant $\Rightarrow y = kx$; $k =$ arbitrary constant that parametrizes the family of characteristics.

Along a characteristic we have $\frac{dx}{x} = \frac{du}{\alpha u} \Rightarrow u = f \cdot x^\alpha$, f = arbitrary constant, but different for each characteristic, so $f = f(k) = f\left(\frac{y}{x}\right)$. So the general solution is

$$u(x,y) = f\left(\frac{y}{x}\right) x^\alpha, \quad f(\cdot) \text{ any } \mathcal{C}^1 \text{ function.}$$

For each $f(\cdot)$, we have an integral surface.

Example:

$$\text{Solve the IVP } \begin{cases} u_t + 2tu_x = 0, \; x \in \mathbb{R}, \; t > 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

Characteristics: $\frac{dt}{1} = \frac{dx}{2t} \Rightarrow dx = 2tdt \Rightarrow x = t^2 + k$, k = arbitrary constant. Along these $\frac{du}{dt} = 0 \Rightarrow u(x,t) = f = \text{const.} = f(k)$. The characteristic through (x,t) starts at $(x,0) = (k,0)$ where $u(k,0) = e^{-k^2}$, so $u(x,t) = u(x,0) = e^{-k^2} = e^{-(x-t^2)}$ is the (unique) integral surface.

Note that because we knew the initial curve: $\Gamma_0 : (x, 0, e^{-x^2})$, $x \in \mathbb{R}$, we were able to find the **unique** integral surface through this curve.

2.C. Solvability of the Cauchy Problem

Given an initial $\mathcal{C}^1$ curve, parametrized by some $s \in I \subset \mathbb{R}$,

$$\Gamma_0 : x = x_0(s), \quad y = y_0(s), \quad z = z_0(s), \quad s \in I \tag{2}$$

we want to find an integral surface $\Sigma : z = u(x,y)$ passing through Γ_0, i.e. satisfying

$$u(x_0(s),\; y_0(s)) = z_0(s), \quad s \in I. \tag{3}$$

The Cauchy data provide us with starting points on Σ for the integral curves, i.e. we now have "initial" conditions for the ODE system that generates the integral curves:

$$\begin{cases} \frac{dx}{d\tau} = a(x,y) \quad , & x(0,s) = x_0(s) \\ \frac{dy}{d\tau} = b(x,y) \quad , & y(0,s) = y_0(s) \\ \frac{du}{d\tau} = c(x,y,u) \quad , & u(0,s) = z_0(s) \end{cases} \tag{7}$$

for each $s \in I$ (just a parameter for this system).

ODE theory tells us: If

$$a,\; b \in \mathcal{C}^1(\Omega'), \Omega^1 = \text{ a neighborhood of } \Gamma_0' \begin{matrix} x = x_0(s) \\ y = y_0(s) \end{matrix}, \quad s \in I$$

$$c \in \mathcal{C}^1 \text{ in a neighborhood } \Omega \text{ of } \Gamma_0$$

then (7) has **unique** solution

$$x = X(\tau, s),\; y = Y(\tau, s),\; u = U(\tau, s) \tag{8}$$

which is $\mathcal{C}^1$ in some neighborhood of $\tau = 0$, $s \in I$. [see Petrovsky p.96: ODE, justify why solution is $\mathcal{C}^1$ $\forall s \in I$].

Now, if we eliminate τ, s in (8), i.e. if we can express τ, s in terms of x, y then we'll get $u(x,y) = U\left(T(x,y),\; S(x,y)\right)$, the solution to our Cauchy Problem. When can we do this? i.e. when can we solve

$$\begin{cases} x = X(\tau, s) \\ y = Y(\tau, s) \end{cases} \tag{9}$$

for τ, s to find $\tau = T(x,y)$, $s = S(x,y)$?

By the Implicit Function Theorem, this can be done in a neighborhood of some point if the Jacobian

$$J := \begin{vmatrix} X_\tau & Y_\tau \\ X_s & Y_s \end{vmatrix} \neq 0 \text{ at that point.} \tag{10}$$

Let's be more precise here. Let s_0 be any point in I. Since X, Y are $\mathcal{C}^1$ in a neighborhood of $(0, s_0)$, by the Implicit Function.(9) has a solution

$$\tau = T(x,y), \quad s = S(x,y) \tag{11}$$

which is $\mathcal{C}^1$ in a neighborhood of $(x_0(s_0),\ y_0(s_0))$ and satisfies

$$0 = T(x_0(s_0),\ y_0(s_0)), \quad s_0 = S(x_0(s),\ y_0(s)). \tag{12}$$

provided

$$J_0 := \begin{vmatrix} X_\tau(0,s_0) & Y_\tau(0,s_0) \\ X_s(0,s_0) & Y_s(0,s_0) \end{vmatrix} \neq 0\,. \tag{13}$$

But

$$J_0 = X_\tau Y_s - X_s Y_\tau \Big|_{\substack{\tau=0\\ s=s_0}} = a(x_0,y_0)y_0'(s) - b(x_0,y_0)x_0'(s)$$

so $J_0 \neq 0$ means

$$a(x_0,y_0)\frac{dy_0}{ds}\Big|_{s=s_0} \neq b(x_0,y_0)\frac{dx_0}{ds}\Big|_{s=s_0}$$

or

$$\frac{dx_0}{a(x_0,y_0)} \neq \frac{dy_0}{b(x_0,y_0)} \text{ at } s = s_0,$$

which means Γ_0' should not be characteristic! We conclude:

Theorem. *If the projection*

$$\Gamma_0' : \begin{cases} x = x_0(s) \\ y = y_0(s) \end{cases}, \quad s \in I$$

of the initial curve Γ_0 on the xy-plane is **not characteristic** *at a point $s = s_0 \in I$, then the Cauchy Problem (1),(2) is solvable in a neighborhood of the point $P(x_0(s_0),\ y_0(s_0),\ z_0(s_0))$.*

Example: Solve the $(CP)\begin{cases} xu_x - yu_y = u^2 \\ u(x,1) = h(x) \end{cases}$

Here $\Gamma_0 :\ x = s,\ y = 1,\ z = h(s)$, and the characteristics are $\frac{dx}{x} = \frac{dy}{-y} \Rightarrow xy = k =$ arbitrary constant, so

$$J_0 = a\,y_0' = bx_0' = x\cdot 0 - (-y)\cdot 1 = +y = 1 \neq 0 \Rightarrow \Gamma_0' : \begin{cases} x = s \\ y = 1 \end{cases}$$

is nowhere characteristic. Solve the ODEs

$$\frac{dx}{d\tau} = x \Rightarrow x = k_1 e^{\tau};\ x(0) = s \Rightarrow x = X(\tau,s) = se^{\tau} \Rightarrow s = \frac{x}{y}$$

$$\frac{dy}{d\tau} = -y \Rightarrow y = k_2 e^{-\tau};\ y(0) = 1 \Rightarrow y = Y(\tau,s) = e^{-\tau} \Rightarrow \tau = -ln\ y$$

$$\frac{du}{d\tau} = u^2 \Rightarrow u = -\frac{1}{\tau + k_3};\ u(0) = h(s) \Rightarrow U(\tau,s) = \frac{h(s)}{1 - \tau\cdot h(s)}$$

$$\Rightarrow u(x,y) = U\left(-ln\ y, \frac{x}{y}\right) = \frac{h\left(\frac{x}{y}\right)}{1 + g\left(\frac{x}{y}\right) ln\ y}$$

is the unique solution, for $y > 0$.

What happens if $J_0 = 0$ on Γ_0', i.e. if Γ_0' is characteristic? Well, either the initial curve Γ_0 itself is an integral curve or it is not. When Γ_0 is an integral curve then taking as initial curve any curve Γ^* such that $J \neq 0$ and Γ^* intersects Γ_0, we can find an integral surface, therefore infinitely many solutions.

When Γ_0 is **not** an integral curve (but Γ_0' is characteristic) then there is no solution through Γ_0 at all.

All of the above can be done with minimal changes even for the quasilinear case, so we can state

Theorem. *Solvability of the Cauchy Problem for Quasilinear 1st order PDE in 2 variables.*

$$(CP) = \begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ u \textit{ passes through } \Gamma_0 : x = x_0(s),\ \ y = y_0(s),\ \ z = u_0(s),\ \ s \in I. \end{cases}$$

If $\dfrac{dx_0(s)}{a(x_0,y_0,u_0)} \neq \dfrac{dy_0(s)}{b(x_0,y_0,u_0)}$ then unique solution.

If $\dfrac{dx_0(s)}{a(x_0,y_0,u_0)} = \dfrac{dy_0(s)}{b(x_0,y_0,u_0)} = \dfrac{du_0(s)}{c(x_0,y_0,u_0)}$ then infinitely many solutions.

If $\dfrac{dx_0(s)}{a(x_0,y_0,u_0)} = \dfrac{dy_0(s)}{b(x_0,y_0,u_0)} \neq \dfrac{du_0(s)}{c(x_0,y_0,u_0)}$ then no solution.