

CHAPTER 0

INTRODUCTION

1. CONCEPTS AND IDEAS

1.A. Basic Concepts

A **PDE** is an equation containing partial derivatives of the unknown which is a function of more than one variables:

$$F(u, u_x, u_y, \dots, u_{xx}, u_{xy}, \dots) = 0.$$

For example, the simplest PDE is $\frac{\partial u(x,y)}{\partial x} = 0$, with obvious solution $u(x,y) = f(y)$ = any function of y .

Issues:

- What do we mean by **solution**? We must be able to plug it into the equation, so the derivatives appearing in the PDE must exist (in some sense).
- For what values of x, y is the equation to hold ? This must be specified as part of the problem.
- What other information is needed to determine a solution? Note that the “general” solution depends on an arbitrary function (not just an arbitrary constant as for an ODE), so additional conditions are always needed.

We summarize:

A **PDE problem** consists of:

- a PDE for an unknown function $u : \mathbb{R}^N \rightarrow \mathbb{R}$
- a region $\Omega \subset \mathbb{R}^N$ where the PDE is to hold (in some sense)
- additional conditions to be satisfied (initial/boundary conditions) coming from the application where the PDE arose.

By **(classical) solution** of a PDE in a region Ω we mean a function u which has those derivatives appearing in the equation continuous in Ω and when substituted into the PDE it satisfies it at each point of Ω . For a PDE problem, the additional conditions must also make sense and be satisfied.

Fundamental Questions:

- Well-posedness of the PDE problem
 - existence: is there a solution ?
 - uniqueness: is there at most one solution ?
 - continuous dependence on data: do small changes in data result in small changes in solutions ?
- can we find explicit solutions? (rather rare)
- properties of solutions: smoothness, maximum principle, etc.

Answers depend on the nature of the constraints imposed on the solution by the PDE, region, subsidiary conditions, and everything depends on the form of the equation itself.

1.B. Useful Forms of PDEs

Since one may write down an infinite variety of PDE's, many of which could be absurd, like $e^{u_x} = 0$ (no solution), we restrict ourselves to **special** forms of equations. Because of their (relative) simplicity and importance in application we'll consider

Linear PDEs:

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x) = 0, \quad x \in \Omega \subset \mathbb{R}^n$$

where

$$\alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \text{ integers}, |\alpha| = \alpha_1 + \dots + \alpha_n, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Quasilinear PDEs:

$$\sum_{|\alpha|=m} a_\alpha(x, u, Du, \dots, D^{m-1}u) D^\alpha u + f(x, u, \dots, D^{m-1}u) = 0$$

(the highest order derivatives appear linearly)

Anything more complicated is called “nonlinear.”

Order of a PDE is the order of the highest derivative appearing.

By far the most important in applications are 2nd order PDEs, linear and quasilinear.

Prototypes:

- Laplace equation: $\Delta u = 0$, the single most important PDE
- Heat or diffusion: $\frac{\partial u}{\partial t} = \alpha \Delta u$, $\alpha = \text{diffusivity} (> 0)$
- Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$, $c = \text{wave speed}$

Other important PDEs: Maxwell equations, ideal gas flow, Navier-Stokes, elastic waves, Schrödinger, Korteweg-de Vries, minimal surfaces, etc.

1.C. What makes the study of PDEs hard

A PDE by itself admits in general, infinitely many solutions (up to arbitrary functions not just up to constants).

The kind of additional conditions needed to make a “well-posed problem” depends **crucially** on the form of the PDE. Even a trivial change, like changing a sign, may alter radically the nature of solutions, so different subsidiary conditions may be required and different methods of study must be applied!

There are PDEs that for no apparent reason **have no solution!!!** The first such striking example was given by Hans Lewy in 1957: a linear equation in 3 variables with infinitely smooth coefficients:

$$[H. Lewy] \quad u_x + 2u_y + 2(ix - y)u_t = f(x, y, t), \quad \text{in the unit ball in } \mathbb{R}^3$$

There are $f \in C^\infty$ for which **no solution exists** (unless f is analytic!). Not only classical solutions, but distributional solutions don’t exist either.

If complex coefficients bother you, this is equivalent to the system

$$\begin{cases} u_x - v_y - 2xv_t - 2yu_t = f_1 \\ v_x + u_y + 2xu_t - 2yv_t = f_2 \end{cases} \quad f_i \text{ real}$$

If you think only systems can be bad, here is a single equation with real C^∞ coefficients:

$$[F. Trèves, 1962] \quad \left[\frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (x^2 + y^2) \frac{\partial^2}{\partial t^2} + \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} \right]^2 u + \frac{\partial^2 u}{\partial t^2} = f$$

that is as bad!!

1.D. The Divergence Theorem (or Gauss Theorem, or Green’s Theorem, or integration by parts)

This is the single most important tool in PDEs. It is a generalization of the Fundamental Theorem of Calculus to many dimensions.

Definition: A **Green domain** in \mathbb{R}^N is a bounded, open, connected $\Omega \subset \mathbb{R}^N$ which admits an outer normal $\mathbf{n} = (n_1, n_2, \dots, n_N)$ on its boundary $\partial\Omega$ and for which the Divergence Theorem holds:

$$\int_{\Omega} \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i ds, \quad i = 1, 2, \dots, N,$$

for any $u \in C^1(\overline{\Omega})$. Here dx = volume element, ds = surface element.

Roughly, a Green domain is a bounded domain with piecewise Lipschitz continuous boundary. Most of the domains that arise in practice are Green domains. In particular a piecewise C^1 domain (its boundary is a finite union of C^1 surfaces) is a Green domain.

Divergence Theorem. *If*

- (i) Ω is a Green domain
- (ii) $u \in C^1(\Omega) \cap C(\overline{\Omega})$
- (iii) $\int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right| dx$ exists (at least as a convergent improper integral)

then

$$\int_{\Omega} \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i ds, \quad i = 1, 2, \dots, N.$$

The name derives from the vector form of the theorem:

Divergence Theorem. *For any Green domain Ω and any vector field \mathbf{U} for which $\nabla \cdot \mathbf{U} \in L^1(\Omega)$, we have*

$$\int_{\Omega} \nabla \cdot \mathbf{U} \, dx = \int_{\partial\Omega} \mathbf{U} \cdot \mathbf{n} \, ds, \quad (\mathbf{n} = \text{outer unit normal to } \partial\Omega).$$

Most general classical treatment appears in Kellogg, *Foundations of Potential Theory*, Dover (1929).

In the case of \mathbb{R}^2 , this is Green's Theorem in the plane:

$$\int_{\Omega} \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right] dx dy = \int_{\partial\Omega} [M n_x + N n_y] ds = \int_{\partial\Omega} M dy - N dx$$

since $n_x ds = dy$, $n_y ds = -dx$.

There is a dictum in modern PDE which says: If you don't know what to do next, integrate by parts:

$$\int_{\Omega} \varphi \frac{\partial u}{\partial x_i} dx \equiv \int_{\Omega} [(\varphi u)_{x_i} - u \varphi_{x_i}] dx = \int_{\partial\Omega} \varphi u n_i ds - \int_{\Omega} u \varphi_{x_i} dx.$$

($\varphi \equiv 1$ is the Divergence Theorem).

2. CONSERVATION LAWS

2.A. The fundamental conservation principles

All of continuum physics (of macroscopic physical processes) is based on three fundamental principles

- (1) Conservation of mass [equation of continuity]
- (2) Conservation of momentum [Newton's 2nd law: force = mass · acceleration]
- (3) Conservation of energy [First Law of Thermodynamics]

These equations in rather general setting (fluid) are:

Mass Conservation: $\rho_t + \nabla \cdot (\rho \mathbf{v}) = f$

ρ = density (mass per unit volume)

\mathbf{v} = velocity

f = source (of mass per volume per time)

Momentum Conservation: $(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + \mathbb{T}) = \rho \mathbf{g}$

\mathbb{T} = stress tensor (surface forces)

\mathbf{g} = body force (e.g. gravity)

Energy conservation: $(\rho \varepsilon)_t + \nabla \cdot (\rho \varepsilon \mathbf{v} + \mathbf{Q}) = \rho E$

ε = total energy per unit mass

\mathbf{Q} = energy flux (non-convective)

E = source (of energy per mass per time)

which must be supplemented by

constitutive laws, relating the fluxes \mathbb{T} , \mathbf{Q} with the state variables (ρ, P, T) , and by an

equation of state, relating the state variables (ρ, P, T) themselves and with ε .

Note that all three laws have the same structure:

$$(\rho u)_t + \nabla \cdot \mathbf{F} = \mathcal{S}$$

ρu = quantity (per unit volume) being conserved

\mathbf{F} = flux of u (quantity per unit area per unit time)

\mathcal{S} = source per volume per time

By extension, any equation of such structure is called a “conservation law.”

This structure results from the basic balance (**conservation**):

amount gained = amount crossing into through the boundaries +
amount created within (during a certain time, or per unit time (rate))

2.B. Integral Balance Law

To express this mathematically for a quantity $u(\mathbf{x}, t)$, which may vary in space and time, let V be an arbitrary but fixed volume $\subset \Omega$ = region of fluid, and

$U(t)$ = total amount of u inside V at any time t

\mathcal{F} = net flow rate through ∂V crossing **into** V

Σ = net amount created inside V per unit time

Then

$$(2) \quad \text{Conservation Principle:} \quad \frac{d}{dt} U = \mathcal{F} + \Sigma$$

But,

$$U = \int_V \rho u \, dx, \quad \rho u = \text{amount per unit volume}, \quad \rho = \text{density}$$

$$\mathcal{F} = - \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, ds,$$

\mathbf{F} = flux = amount per unit area per unit time,

\mathbf{n} = outgoing unit normal to ∂V

$$\Sigma = \int_V \mathcal{S} \, dx, \quad \mathcal{S} = \text{source density} = \text{amount created per unit volume per unit time.}$$

whence we obtain the

$$(3) \quad \text{Conservation Principle:} \quad \frac{d}{dt} \int_V \rho u \, dx = - \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, ds + \int_V \mathcal{S} \, dx$$

Note that this makes sense as long as the **integrals** exist, since it involves only **averages**, which is all that can be actually observed and measured ! Both physically and mathematically, this is the most general

form of conservation, and the concept of weak solution is based directly on it, and so are some of the best numerical schemes: $\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\sum A\mathcal{F}_{i-\frac{1}{2}} + \mathcal{S}_i$. Note that ρu , $\mathbf{F} \cdot \mathbf{n}$, \mathcal{S} may even be discontinuous in this setting !

2.C. Differential form of Conservation Principle

For conceptual and operational purposes, it is more convenient to localize the law. The operation of “divergence” was invented for this purpose, and the Divergence Theorem changes the surface integral in (3) into a volume integral that can be combined with the rest. Thus, assuming $\rho(\mathbf{x}, t)$, $u(\mathbf{x}, t)$, and $\mathbf{F}(\mathbf{x}, t)$ are defined **pointwise** and are **smooth** (enough for the Divergence Theorem to hold), we can write (3) as

$$(4) \quad \int_V \left\{ \frac{\partial(\rho u)}{\partial t} + \nabla \cdot \mathbf{F} - \mathcal{S} \right\} d\mathbf{x} = 0 \quad \text{for any volume } V$$

for which the Divergence Theorem holds. Note that the divergence operation $\nabla \cdot (\)$ is with respect to space variables only. The arbitrariness of the volume allows us to conclude the localized form of the conservation principle:

$$(5) \quad \frac{\partial}{\partial t}(\rho u) + \nabla \cdot \mathbf{F} = \mathcal{S} \quad \text{at each } (\mathbf{x}, t).$$

Note that (5) **assumes smoothness**, so it is **not** as general as the integral form (3). The fundamental conservation laws listed in §2.A are obtained as follows.

Conservation of species:

To conserve the mass of species i in a mixture with flow, diffusion, and chemical reactions present, let

$$\begin{aligned} u &:= C^i = \text{mass fraction of species } i \text{ in the mixture} \\ &= \frac{(\text{mass of } i)/V}{(\text{mass of mixture})/V} = \frac{(\text{partial) density of } i}{\text{total density}} = \frac{\rho^i}{\rho}, \end{aligned}$$

so that $\rho u = \rho C^i = \rho^i$ is the mass of i per unit volume. The total flux is $\mathbf{F} = \rho^i \mathbf{v}^i$ with \mathbf{v}^i the **velocity of species i** , which we can reference to the mixture velocity \mathbf{v} by writing $\mathbf{F} = \rho^i \mathbf{v} + \rho^i (\mathbf{v}^i - \mathbf{v}) = \rho^i \mathbf{v} + \mathbf{j}^i$, with $\mathbf{j}^i := \rho^i (\mathbf{v}^i - \mathbf{v}) = \text{diffusion flux of species } i$ (to be specified by a constitutive law). Then (5) yields

$$(6) \quad \rho_t^i + \nabla \cdot [\rho^i \mathbf{v} + \mathbf{j}^i] = R^i,$$

with $R^i :=$ rate at which i is produced by chemical reactions. In terms of the mass fraction $C^i = \rho^i/\rho$, (6) reads

$$(7) \quad (\rho C^i)_t + \nabla \cdot [(\rho C^i) \mathbf{v} + \mathbf{j}^i] = R^i.$$

Equation of continuity: (for the mixture)

Clearly, $\sum_i \rho^i \equiv \sum_i \rho C^i = \rho$ since $\sum_i C^i = 1$. Also, $\sum_i \rho^i \mathbf{v}^i = \rho \mathbf{v}$ since \mathbf{v} is the (average) mixture velocity. Hence $\sum_i \mathbf{j}^i = \sum_i \rho^i (\mathbf{v}^i - \mathbf{v}) = \rho \mathbf{v} - \rho \mathbf{v} = 0$, i.e. the total diffusion flux is zero. Moreover, conservation of total mass dictates $\sum_i R^i = 0$. Summing up (6) over all the species constituting the mixture, we obtain the continuity equation:

$$(8) \quad \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0.$$

Conservation of momentum: (equation of motion)

To conserve momentum $:= \rho \mathbf{v}$, take $u = \mathbf{v}$ in (5). The momentum flux is the **convective flux** $(\rho \mathbf{v}) \mathbf{v}$ (a matrix with ij -th entry $\rho v_i v_j$) plus the **stress tensor** \mathbb{T} (forces acting on the boundary of a volume). The source term is $\mathcal{S} := \rho \mathbf{g}$ if gravity is the only body force. Then (5) yields

$$(9) \quad (\rho \mathbf{v})_t + \nabla \cdot [(\rho \mathbf{v}) \mathbf{v} + \mathbb{T}] = \rho \mathbf{g},$$

expressing Newton's second law of motion.

Conservation of energy:

To conserve total energy (internal + kinetic), take $u := \varepsilon$ in (5). The energy flux is the **convective flux** $(\rho \varepsilon) \mathbf{v}$ plus the sum of all non-convective fluxes of energy (due to conduction, interdiffusion, radiation, etc.), which we denote by \mathbf{Q} . Writing ρE for any internal heat sources, (5) yields

$$(10) \quad (\rho \varepsilon)_t + \nabla \cdot [(\rho \varepsilon) \mathbf{v} + \mathbf{Q}] = \rho E.$$

Advective form of conservation laws:

These laws may be written in many other forms, depending on the choice of primary variables, frames of reference, mass or molar units, etc. An excellent reference is Bird-Stewart-Lightfoot, Transport Phenomena, Wiley, 1960.

Using the continuity equation, (7), (9), (10) are often expressed in a somewhat simpler “advective form” (whereas (7) - (10) are in “divergence form”). Note that their structure is

$$(11) \quad (\rho u)_t + \nabla \cdot [(\rho u) \mathbf{v}] = -\nabla \cdot \Phi + \rho \mathcal{S}.$$

Using the product rule, the left hand side breaks up into $\rho_t u + \rho u_t + u \nabla \cdot [\rho \mathbf{v}] + \rho \mathbf{v} \cdot \nabla u$ which, thanks to (8), simplifies to $\rho [u_t + \mathbf{v} \cdot \nabla u]$. Hence, (11) takes the “advective form”

$$(12) \quad u_t + \mathbf{v} \cdot \nabla u = -\frac{1}{\rho} \nabla \cdot \Phi + \mathcal{S}.$$

Note that the left-hand side is the total time derivative of u in a Lagrangian frame, i.e. for an observer moving along a flow line with velocity \mathbf{v} . Indeed, by the chain rule, for an observer at $\mathbf{x}(t)$ (where $\frac{d\mathbf{x}}{dt} = \mathbf{v}$) we have

$$\frac{d}{dt} u(\mathbf{x}(t), t) = \nabla u \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial u}{\partial t} = \mathbf{v} \cdot \nabla u + u_t.$$

Hence, the term $\mathbf{v} \cdot \nabla u$ measures the change due to *advection* with velocity \mathbf{v} , whence it is called the **advective term**. The operation

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

is called the **material derivative**, and (12) may be written simply as

$$\frac{Du}{Dt} = -\frac{1}{\rho} \nabla \cdot \Phi + \mathcal{S}.$$

The advective forms of (7), (9), (10) are

$$(7') \quad C_t^i + \mathbf{v} \cdot \nabla C^i = -\frac{1}{\rho} \nabla \cdot \mathbf{j}^i + \frac{1}{\rho} R^i$$

$$(9') \quad \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \cdot \mathbb{T} + \mathbf{g}$$

$$(10') \quad \varepsilon_t + \mathbf{v} \cdot \nabla \varepsilon = -\frac{1}{\rho} \nabla \cdot \mathbf{Q} + E$$

In (9'), the advective term $(\mathbf{v} \cdot \nabla) \mathbf{v}$ is a vector with components $\sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j}$ (in 3-dimensions). In 1-dimension, it is vv_x .

3. IMPORTANT EQUATIONS–PROCESSES

We present the equations arising in some of the most important physical processes.

3.A. Diffusion:

The relative movement of molecules of one substance into another is called **diffusion** or **interdiffusion**. It is one of the most fundamental physical phenomena, operating in every living cell and all around us.

Let $C(x, t)$ be the mass fraction (§2.C) of a diffusing species at location x at time t . The simplest, and eminently successful, constitutive law for diffusion is

$$(13) \quad \textbf{Fick's Law : } \quad \mathbf{j} = -\rho D \nabla C, \quad D = \text{diffusivity}(\text{cm}^2/\text{sec}).$$

The diffusivity D (also called *diffusion coefficient*) depends, in general, on concentration, temperature, and pressure, and of course on the materials involved, so possibly also on location. Then (7) becomes a quasilinear 2^{nd} order PDE called the

$$(14) \quad \textbf{advection-diffusion-reaction : } \quad (\rho C)_t + \nabla \cdot [\rho C \mathbf{v} - \rho D \nabla C] = R.$$

If there is no flow ($\mathbf{v} \equiv \mathbf{0}$) and $\rho \equiv \text{constant}$, it reduces to the (quasilinear)

$$(15) \quad \textbf{diffusion-reaction equation: } \quad C_t = \nabla \cdot (D \nabla C) + \frac{1}{\rho} R,$$

and if $D \equiv \text{constant}$, it simplifies to the semilinear equation ($\nabla \cdot \nabla = \nabla^2 = \Delta$, the Laplacian)

$$(16) \quad C_t = D \nabla^2 C + \frac{1}{\rho} R(C).$$

Finally, if the reaction (source) term is independent of C , we have the **linear diffusion equation**:

$$(17) \quad C_t = D \nabla^2 C + f(x, t).$$

Rescaling time to dimensionless diffusion time: $\tau = \frac{D}{\ell^2} t$, $\ell = \text{some length scale}$, we arrive at the prototype of linear 2^{nd} order **parabolic** equations, commonly called the **Heat** or **Diffusion equation**:

$$(18) \quad u_t = \nabla^2 u + f.$$

The one-dimensional versions of (14), (16), (18) are (for $\rho \equiv \text{const.}$, for some source f):

$$(14') \quad C_t + (vC)_x = (DC_x)_x + f$$

$$(16') \quad C_t = DC_{xx} + f$$

$$(18') \quad u_t = u_{xx} + f.$$

Steady-state (time-independent) diffusion leads to the steady-state versions of the above (with $C_t \equiv 0$), which are **elliptic-type** PDEs, e.g.

$$(14'') \quad \nabla \cdot [\rho C \mathbf{v} - \rho D \nabla C] = R,$$

$$(18'') \quad \textbf{Poisson equation : } \quad \Delta u = f$$

$$\textbf{Laplace equation : } \quad \Delta u = 0, \text{ prototype of linear 2nd order elliptic equations.}$$

3.B. Two-phase flow in porous media:

Consider the flow of two incompressible immiscible fluids (oil and water) in soil. Let ϕ =porosity, ρ_α =density of phase $\alpha = o, w$, S_α =saturation of phase α (fraction of pore volume occupied by phase α), $S_o + S_w = 1$, u_α =filtration (Darcy) velocity ($= \phi \mathbf{v}$), P_α =pressure, with $P_o - P_w = P_{cap}(S_w)$ =capillary pressure. Then, ignoring dispersion, conservation of α is described by:

$$\begin{aligned} \text{advection: } & (\phi \rho_\alpha S_\alpha)_t + \nabla \cdot (\rho_\alpha u_\alpha) = q_\alpha, \quad \alpha = o, w \\ \text{Darcy Law : } & u_\alpha = -\lambda_\alpha (\nabla P_\alpha - \rho_\alpha g), \quad \alpha = o, w \end{aligned}$$

where $\lambda_\alpha = \text{mobility} = K_\alpha / \mu_\alpha$, K_α =permeability, μ_α =viscosity, g =gravity. These are six equations for the unknowns $S_\alpha, u_\alpha, P_\alpha$, which can be reduced to two equations, a continuity equation for S_w and an elliptic equation for the total pressure $P = P_o + P_w$, referred to as *pressure-saturation* formulation.

3.C. Contaminant transport in soil:

Let ρ = concentration of a pollutant in groundwater, N = concentration of sorbed pollutant in saturated soil of porosity ϕ ($\equiv \text{const.}$) transported by water flowing with filtration velocity u . Ignoring gravity, the process is described by:

$$\begin{aligned} \text{advection-dispersion: } & \phi \rho_t + \nabla \cdot (\rho u - D \nabla \rho) = N_t, \quad D = \text{dispersivity coefficient} \\ \text{dissorption: } & N_t = \alpha \rho - \beta N, \quad \alpha > 0, \beta > 0 \quad (\text{simplest model}) \\ \text{Darcy Law : } & u = -\lambda \nabla P, \quad \lambda = \text{mobility, as above, } P = \text{pressure} \\ \text{continuity: } & \nabla \cdot (\lambda \nabla P) = \phi \rho_t \approx 0 \quad \text{or } \Delta P = 0 \quad (\text{elliptic equation for pressure}). \end{aligned}$$

3.D. Conduction of heat:

Equation of state: $d\varepsilon = c_p dT$; T = temperature, c_p = heat capacity $= c_p(T)$. In the absence of convective effects, as e.g. in a solid, $\mathbf{v} \equiv 0$. Conduction is characterized by the most famous constitutive law:

$$\text{Fourier's law : } \mathbf{Q} = -k \nabla T, \quad k(T) = \text{thermal conductivity}.$$

Thus, energy conservation (10) leads to

$$(19) \quad \rho c_p T_t = \nabla \cdot (k \nabla T) + f : \quad \text{heat conduction equation}$$

If $k \equiv \text{const.}$, set $\alpha = \frac{k}{\rho c_p}$ = thermal diffusivity (cm^2/sec)

$$(20) \quad T_t = \alpha \nabla^2 T + f \quad : \quad (\text{linear}) \text{ heat equation}$$

Rescaling time to dimensionless *Fourier Number*: $\tau := \frac{\alpha}{\ell^2} t$:

$$(18) \quad u_\tau = \nabla^2 u + f \quad (\text{model parabolic PDE}).$$

$$(18'') \quad \text{Steady-state heat conduction : } \nabla^2 u = f \quad (\text{model elliptic PDE}).$$

Note that both diffusion and heat conduction are described by the **same** equation, whence we can think of heat as “diffusing”!

The simplest forms of (18) in 1, 2 and 3 dimensions, without sources, are

$$\begin{aligned} u_t &= u_{xx} \\ u_t &= u_{xx} + u_{yy} = \Delta_{x,y} u \\ u_t &= u_{xx} + u_{yy} + u_{zz} = \Delta_{x,y,z} u \end{aligned}$$

2-dimensional in polar coordinates: $u(r, \theta, t) : u_t = \Delta_{r,\theta} u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$

3-dimensional axially symmetric: $u(r, z, t) : u_t = \Delta_{r,z} u = \frac{1}{r}(ru_r)_r + u_{zz}$

3-dimensional in spherical coordinates $u(r, \theta, \varphi, t) : u_t = \frac{1}{r^2}(r^2u_r)_r + \frac{1}{r^2 \sin \theta}(\sin \theta u_\theta)_\theta + \frac{1}{r^2 \sin^2 \theta}u_{\varphi\varphi}$

3.E. Vibrating string: [W. Strauss, p.11]

Consider a flexible, elastic, homogeneous string undergoing *small* transverse vibrations. Let $u(x, t)$ be the vertical displacement from equilibrium, $\rho_0 =$ density (mass/length) \approx constant (due to being homogeneous), $T(x, t) =$ magnitude of tension vector, tangential to string (flexible).

Then $\frac{T}{\sqrt{1+u_x^2}} = \text{const.} =: \tau$, so $\tau(u_x|_{x+\Delta x} - u_x|_x) \frac{1}{\Delta x} = \frac{1}{\Delta x} \int_x^{x+\Delta x} \rho_0 u_{tt} dx$. Setting $c^2 := \tau/\rho_0$:

$$u_{tt} = c^2 u_{xx}, \quad \text{the **wave equation**, prototype of linear hyperbolic PDEs.}$$

If there is friction (opposing the motion), assumed proportioned to the velocity u_t , then

$$\text{damped wave equation : } u_{tt} + bu_t = c^2 u_{xx}.$$

3.F. Gas dynamics: - flow of a perfect inviscid fluid

$$\mathbb{T} \approx +P \mathbb{1} \approx \quad (\text{no viscosity}), \quad \mathcal{S} = \rho \mathbf{g} \approx 0$$

\Rightarrow Euler equations for isentropic inviscid flow

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P \\ \text{equ. of state : } P = f(\rho) : \frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \text{ for polytropic gas, } \gamma = \frac{c_p}{c_v} > 1 \end{cases}$$

Acoustic approximation: Assume small $u := \frac{\rho - \rho_0}{\rho_0}$, so $\rho = \rho_0(1 + u)$, and small v so that all products ≈ 0 . Then $P \approx P_0 + f'(\rho_0)(\rho - \rho_0) = f(\rho_0) + f'(\rho_0)\rho_0 u + O(\rho^2)$

$$\begin{aligned} \Rightarrow u_t + \nabla \cdot \mathbf{v} &= 0, \quad \rho_0(1 + u) [\mathbf{v}_t + \nearrow^0] \approx -f'(\rho_0)\rho_0 \nabla u \mathbf{v}_t \approx -f'(\rho_0) \nabla u \\ \Rightarrow \begin{cases} u_t + \nabla \cdot \mathbf{v} = 0 & \text{Eliminate } \mathbf{v} : u_{tt} + \nabla \cdot \mathbf{v}_t = 0 \\ \mathbf{v}_t + f'(\rho_t) \nabla u = 0 & \Rightarrow u_{tt} - \nabla \cdot (f'(\rho_0) \nabla u) = 0 \end{cases} \end{aligned}$$

Setting $c^2 = f'(\rho_0)$ ($= \frac{\gamma P_0}{\rho_0}$ for polytropic gas), we get the

$$\text{wave equation : } u_{tt} - c^2 \Delta u = 0,$$

prototype of linear hyperbolic equations, by which signals propagate with **finite** speed c .

$$\text{A more general system is: } \begin{cases} u_t + v_x = 0 \\ v_t + (F(u))_x = 0 \end{cases}$$

Eliminating v : $u_{tt} - (F(u))_{xx} = 0$ (nonlinear), or setting $u = w_x$, $v = w_t$:

$$\begin{aligned} w_{tt} &= v_t = -F'(w_x)w_{xx} \\ \Rightarrow w_{tt} + F'(w_x)w_{xx} &= 0 \quad \text{quasilinear, hyperbolic for } F' < 0, \text{ elliptic for } F' > 0 \end{aligned}$$

3.G. Navier-Stokes equations:

For Newtonian fluid: $\mathbb{T} = P\mathbf{1} - \Pi$, with viscous stress tensor:

$$\Pi = \lambda(\operatorname{div} \mathbf{v})\mathbf{1} + 2\mu D \quad (\text{skew part} + \text{symmetric part}), \quad D = \text{deformation tensor}$$

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\text{compressible: } (\rho \mathbf{v})_t + \nabla \cdot [(\rho \mathbf{v}) \mathbf{v}] = \rho \mathbf{g} - \nabla P + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{v}) + \mu \Delta \mathbf{v}$$

$$\text{incompressible : } (\rho \equiv \text{const.}) \begin{cases} \mathbf{v}_t + \nabla \cdot (\mathbf{v} \mathbf{v}) = \mathbf{g} - \frac{1}{\rho} \nabla P + \nu \Delta \mathbf{v} \\ \operatorname{div} \mathbf{v} = 0 \end{cases} \quad \nu = \frac{\mu}{\rho} = \text{kinematic viscosity}$$

3.H. Hyperbolic Heat Conduction: (1-dimensional)

$\rho c_p T_t + Q_x = 0$, but *instead* of Fourier's law, we have: $\tau Q_t + Q = -k T_x$ with $\tau = \text{"delay time"}$, which follows from the constitutive law $Q(t + \tau) = -k T_x$, by approximating $Q(t + \tau) \approx Q + \tau Q_t$.
Eliminating Q : $\rho c_p T_{tt} + Q_{xt} = 0$, $\tau Q_{xt} + Q_x = -k T_{xx} \Rightarrow \rho c_p T_{tt} - \frac{k}{\tau} T_{xx} + \frac{\rho c_p}{\tau} T_t = 0$.
Hence $T_{tt} + \frac{1}{\tau} T_t = \frac{\alpha}{\tau} T_{xx}$, Telegrapher's equation, which has finite propagation speed.

3.I. Seismic wave parabolic approximation: $P_z = (i\omega/v)P + (iv/\omega^2)P_{xx}$,

for a downgoing plane pressure wave $P(x, z, \omega)$ of frequency ω , with $v(x, z) = \text{"lateral velocity variation"}$ at lateral location x and depth z .

[Claerbout, Basic Earth Imaging, 2000 (draft book: sepwww.stanford.edu/sep/prof/)]

3.J. Equation of linear elasticity:

Steady state: $\Delta^2 u = 0$ (biLaplacian)

3.K. Schrödinger equation: (quantum mechanics)

$$i\hbar u_t = -\left(\frac{\hbar^2}{2m}\right) \Delta u + V u, \quad \int_{\Omega} |u(\mathbf{x}, t)|^2 = \text{probability particle is in } \Omega \text{ at time } t.$$

steady state: $\Delta u + V u = 0$, $V = \text{potential}$

3.L. Hamilton-Jacobi equations: $u_t + H(u, \nabla u) = 0$, $H = \text{"Hamiltonian"}$ (see Kevorkian)

In optimal control theory: $u_t + \max_{\alpha} (\mathbf{b}^\alpha \cdot \nabla u + c^\alpha u) = 0$.

3.M. Euler-Lagrange: (variational) $u_t = \frac{\delta J(u)}{\delta u} = \frac{dJ(\varepsilon u)}{d\varepsilon} \big|_{\varepsilon=0}$

3.N. KdV equation: (a simple form): $u_t + uu_x + u_{xxx} = 0$ arises in "swallow water waves."

It admits traveling wave solutions called solitons: $u(x, t) = s(x - ct)$, $s(x) = 3c \operatorname{sech}^2\left(\frac{x\sqrt{c}}{2}\right)$.

3.O. Maxwell Equations: for electric field strength \mathbf{E} , magnetic field \mathbf{B}

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} & \rho &= \text{density of free charges} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \varepsilon_0 &= \text{dielectric constant} \\ \nabla \cdot \mathbf{B} &= 0 & c &= \text{speed of light in vacuum} \\ c^2 \nabla \mathbf{B} &= \frac{\sigma}{\varepsilon_0} \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t} & \sigma &= \text{electrical conductivity} \end{aligned}$$

From these one derives [Guenther-Lee p.492], [Tichonov-Samarski p.492] the wave equations

$$\begin{aligned} c^2 \Delta \varphi - \frac{\partial^2 \varphi}{\partial t^2} &= -\frac{c^2}{\varepsilon_0} \rho \\ c^2 \Delta \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{1}{\varepsilon_0} \mathbf{j} \end{aligned}$$

for the electric potential φ : $\mathbf{E} = -\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t}$, and vector magnetic potential \mathbf{A} : $\mathbf{B} = \nabla \times \mathbf{A}$.