Explicitly solvable advection-diffusion problems

1. Constant $\vec{v}$ adv-diff can be reduced to diffusion

A. $u(x,t)$ solves $u_t + Vu_x = (Du_x)_x$ iff

$w(\xi,t) = u(x,t)$ with $\xi = x-(x_0+Vt)$ solves $w_t = (Dw_\xi)_\xi$

for $V =$ const., x_0 const.

So if we can solve the pure diff equ, then can solve adv-diff.

Proof: $u_x = w_\xi \cdot \xi_x = w_\xi$, $u_{xx} = w_{\xi\xi}$, $u_t = w_t + w_\xi \cdot \xi_t = w_t + w_\xi \cdot (-V)$

so $u_t + Vu_x = (Du_x)_x \Rightarrow w_t - Vw_\xi + Vw_\xi = \frac{\partial}{\partial \xi}\left(D \frac{\partial w}{\partial \xi} \cdot 1\right) \cdot 1 = (Dw_\xi)_\xi$

B. Same holds in any dimension: for const. $\vec{v}$, $w(\vec{\xi},t) = u(\vec{x},t)$

$$u_t + \vec{v}\cdot\nabla u = \nabla\cdot(D\nabla u) \underset{\vec{\xi} = \vec{x} - \vec{v}t}{\Longleftrightarrow} w_t = \nabla_\xi \cdot (D \nabla_\xi w)$$

i.e. $\xi_i = x_i - v_i t$, $i = 1,2,\ldots$

Proof: $\frac{\partial}{\partial x_i}(\;) = \frac{\partial}{\xi_i}(\;) \cdot \cancelto{1}{\frac{\partial \xi_i}{\partial x_i}} \Rightarrow u_{x_i} = w_{\xi_i}$, $i=1,2,\ldots \Rightarrow \nabla_x u = \nabla_\xi w$.

$$u_t = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial \xi_1}\cdot\frac{\partial \xi_1}{\partial t} + \frac{\partial w}{\partial \xi_2}\cdot\frac{\partial \xi_2}{\partial t} + \ldots = w_t + \left\langle \frac{\partial w}{\partial \xi_1}, \frac{\partial w}{\partial \xi_2}, \ldots \right\rangle \cdot \langle -v_1, -v_2, \ldots \rangle$$

$$= w_t + \nabla w \cdot (-\vec{v})$$

so u-PDE $\Rightarrow w_t - \cancel{\vec{v}\cdot\nabla w} + \cancel{\vec{v}\cdot\nabla w} = \nabla_\xi \cdot (D \nabla_\xi w)$

$\Rightarrow w_t = \nabla_\xi \cdot (D \nabla_\xi w)$

Dirac δ "function" it is not a function!

it is a continuous linear functional on "test" functions

distributions or generalized functions : $T : \mathcal{D} \to \mathbb{R}$ or $\mathbb{C}$, $\quad \mathcal{D} = C_0^\infty(\Omega)$

T acts on a test fn φ and produces a number,
just like an integral does (averaging)
$T(\varphi) \in \mathbb{R}$; but linear, std notation : $\langle T, \varphi \rangle \equiv T(\varphi)$

e.g. $\langle T_f, \varphi \rangle = \int_\Omega f(x)\,\varphi(x)\,dx$ for f integrable on Ω
is a "regular" distribution

Dirac δ : $\langle \delta, \varphi \rangle := \varphi(0) \quad \forall \varphi \in \mathcal{D}$, $\quad$ "$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx$" $= \varphi(0)$

. If it were a function would need to satisfy
$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \quad \text{but with } \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$
! no such function!

Used in Physics for decades symbolically, finally given math meaning in 1948 by Laurent Schwarz, as functional on test functions.

Has Fourier Transform $\mathcal{F}[\delta] = 1$, so $\mathcal{F}^{-1}[1] = 2\pi\,\delta$

Fourier Transform: $\mathcal{F}[f(x)](\xi) = \int_{-\infty}^{\infty} f(x)\,e^{-ix\xi}\,dx =: \hat{f}(\xi)$

Inverse Fourier Transform: $\mathcal{F}^{-1}[\hat{f}(\xi)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\,e^{+ix\xi}\,d\xi = f(x)$

$\mathcal{F}[f'(x)](\xi) = i\xi\,\hat{f}(\xi)$

Convolution property: $\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$

Linear operator: $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \quad \forall$ scalars α, β
"sums split and constants come out"

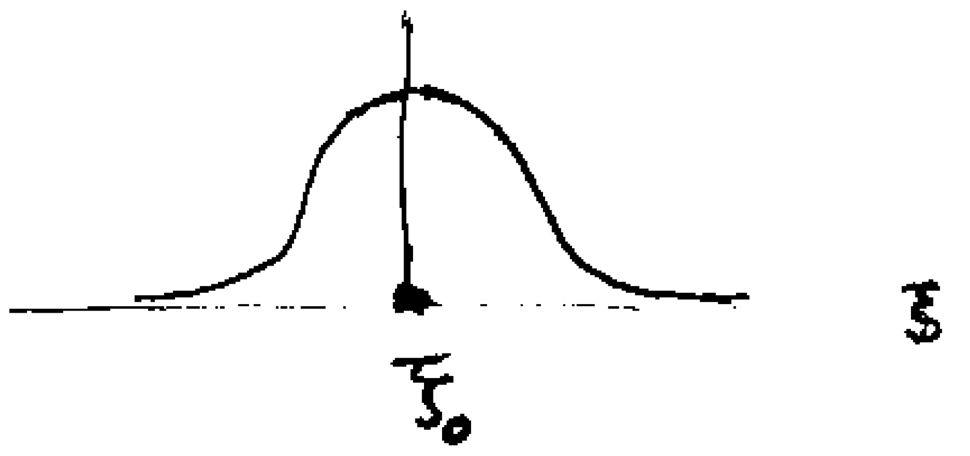
e.g. $L(\vec{x}) = A\vec{x}$, A $m \times n$ matrix, $L : \mathbb{R}^n \to \mathbb{R}^m$
Any linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as $A\vec{x}$ with $A_{m \times n}$

e.g. $L(f) = \int_a^b f(x)\,dx$, $L(f) = \frac{df}{dx}$ are linear!

Some simple solutions for 1D diffusion and linear advection diffusion IVP

3. Point source initially: $\begin{cases} w_t = D w_{\xi\xi}, & -\infty<\xi<\infty,\ t>0 \\ w(\xi,0) = M\,\delta(\xi-\xi_0) \end{cases}$

Dirac δ at $\xi=\xi_0$



has solution the Gaussian

$$w(\xi,t) = \underbrace{\frac{M}{\sqrt{4\pi D t}}}_{\sqrt{4\pi Dt}\,=\,\sigma\sqrt{\pi}}\, e^{-\frac{(\xi-\xi_0)^2}{\underbrace{4Dt}_{\sigma^2}}} \quad = \text{bell curve centered at } \xi=\xi_0 \text{ with std deviation } \sigma = 2\sqrt{Dt}$$

$$= M\cdot H(\xi-\xi_0,\, t-t_0) \quad = \text{heat kernel}$$

Proof: via Fourier transform: Let $\hat{w}(\eta,t) = \mathcal{F}[w(\xi,t)](\eta) = \int_{-\infty}^{\infty} w(\xi,t)\, e^{-i\xi\eta}\, d\xi$

Take FT: $\hat{w}_t = D\eta^2 \hat{w}$, $\hat{w}(0) = M\cdot e^{-i\eta\xi_0} \Rightarrow \hat{w}(\eta,t) = Me^{-D\eta^2 t}\cdot e^{-i\eta\xi_0}$

Take IFT: $w(\xi,t) = \mathcal{F}^{-1}[\hat{w}(\eta,t)](\xi) = \frac{M}{2\pi}\int_{-\infty}^{\infty} e^{-Dt\eta^2} e^{+i(\xi-\xi_0)\eta}\, d\eta$ ∴ set $\alpha = Dt$, $\bar{\xi} = \xi-\xi_0$

exponent $= -\alpha\eta^2 + i\bar{\xi}\eta = -\alpha\left[\eta^2 - 2\frac{i\bar{\xi}}{2\alpha}\eta + \left(\frac{i\bar{\xi}}{2\alpha}\right)^2 - \left(\frac{i\bar{\xi}}{2\alpha}\right)^2\right] = -\alpha\left(\eta - \frac{i\bar{\xi}}{2\alpha}\right)^2 - \frac{\bar{\xi}^2}{4\alpha}$

$$\Rightarrow w(\xi,t) = \frac{M}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\bar{\xi}^2}{4\alpha}} e^{-\alpha\left(\eta-\frac{i\bar{\xi}}{2\alpha}\right)^2} d\eta = M\frac{e^{-\frac{\bar{\xi}^2}{4\alpha}}}{2\pi\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-z^2}\, dz = M\frac{e^{-\frac{\bar{\xi}^2}{4\alpha}}}{2\pi\sqrt{\alpha}}\sqrt{\pi} = \frac{M}{\sqrt{4\pi Dt}}\, e^{-\frac{(\xi-\xi_0)^2}{4Dt}}$$

$z = \sqrt{\alpha}\left(\eta - \frac{i\bar{\xi}}{2\alpha}\right) \Rightarrow d\eta = \frac{1}{\sqrt{\alpha}}\, dz$

a Gaussian!

Therefore sol of advection-diffusion with **IC**: $u(x,0) = M\delta(x-x_0)$ is $(\xi = x-x_0-Vt)$

$$u(x,t) = \frac{M}{\sqrt{4\pi Dt}}\, e^{-\frac{(x-x_0-Vt)^2}{4Dt}}$$

2. **Fourier–Poisson Integral:** $\begin{cases} w_t = D w_{\xi\xi}, & -\infty<\xi<\infty,\ t>0 \\ w(\xi,t_0) = w_0(\xi) \end{cases}$

has solution

$$w(\xi,t) = \int_{-\infty}^{\infty} w_0(\eta)\, H(\xi-\eta,\, t-t_0)\, d\eta, \quad \text{where } H(\xi,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-\frac{\xi^2}{4Dt}} \text{ heat kernel}$$

$$= H * w_0 \quad \text{convolution (in space)}$$

e.g. For $w_0(\xi) = M\cdot\delta(\xi-\xi_0)$, $t_0 = 0$: $w(\xi,t) = \int_{-\infty}^{\infty} M\,\delta(\eta-\xi_0)\, H(\xi-\eta,\, t-0)\, d\eta = M\cdot H(\xi-\xi_0,\, t)$

as found above

4. Gaussian initially: $\begin{cases} w_t = D w_{\xi\xi}, & -\infty < \xi < \infty, t>0 \\ w(\xi,0) = e^{-\frac{(\xi-\xi_0)^2}{\sigma}} =: w_0(\xi) \end{cases}$

has solution $w(\xi,t) = \sqrt{\frac{\sigma}{\sigma+4Dt}}\, e^{-\frac{(\xi-\xi_0)^2}{\sigma+4Dt}}$

Proof: Fourier-Poisson integral gives as solution

$$w(\xi,t) = \int_{-\infty}^{\infty} w_0(\eta) H(\xi-\eta,t)\,d\eta = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(\xi-\eta)^2}{4Dt}} \cdot e^{-\frac{(\eta-\xi_0)^2}{\sigma}}\,d\eta$$

which we can compute directly by algebra on exponent and integration

or, observing it is a convolution of H and w_0, can take FT and then invert:

$\mathcal{F}[f(x)] = \int f(x) e^{-i\omega x}\,dx$, $\mathcal{F}^{-1}[\hat{f}] = \frac{1}{2\pi}\int \ldots$

FT formula for gaussians: $\mathcal{F}[e^{-ax^2}](\eta) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\eta^2}{4a}}$, $\mathcal{F}^{-1}[e^{-b\omega^2}](x) = \frac{1}{\sqrt{4\pi b}}\, e^{-\frac{x^2}{4b}}$

FT of convolution: $\mathcal{F}[H * w_0] = \mathcal{F}[H]\,\mathcal{F}[w_0]$

$$\mathcal{F}[H] = \frac{1}{\sqrt{4\pi Dt}}\sqrt{\frac{\pi}{\frac{1}{4Dt}}}\, e^{-\frac{\eta^2}{\cancel{4}\frac{1}{\cancel{4}Dt}}} = e^{-Dt\eta^2}, \quad \mathcal{F}[w_0(\xi)](\eta) = \sqrt{\pi\sigma}\, e^{-\frac{\sigma}{4}\eta^2}$$

$$\therefore \hat{w}(\eta,t) = \mathcal{F}[H * w_0] = \mathcal{F}[H]\cdot\mathcal{F}[w_0] = e^{-Dt\eta^2}\cdot\sqrt{\pi\sigma}\, e^{-\frac{\sigma\eta^2}{4}} = \sqrt{\pi\sigma}\, e^{-(Dt+\frac{\sigma}{4})\eta^2} = \sqrt{\pi\sigma}\, e^{-\frac{(\sigma+4Dt)}{4}\eta^2}$$

$$\Rightarrow w(\xi,t) = \mathcal{F}^{-1}[\hat{w}(\eta,t)](\xi) = \sqrt{\pi\sigma}\,\mathcal{F}^{-1}[e^{-b\eta^2}] \text{ with } b = \frac{\sigma+4Dt}{4}$$

$$= \sqrt{\cancel{\pi}\sigma}\cdot\frac{1}{\sqrt{4\cancel{\pi}b}}\, e^{-\frac{\xi^2}{4b}} = \sqrt{\frac{\sigma}{\sigma+4Dt}}\, e^{-\frac{(\xi-\xi_0)^2}{\sigma+4Dt}}$$

Sol of advection-diffusion with initial gaussian: $u(x,0) = M e^{-\frac{(x-x_0)^2}{\sigma}}$

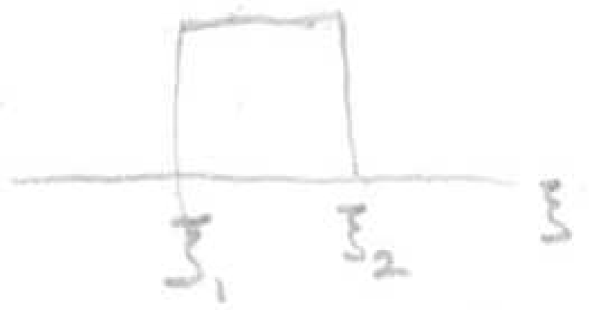
is $u(x,t) = M\sqrt{\frac{\sigma}{\sigma+4Dt}}\, e^{-\frac{(x-x_0-Vt)^2}{\sigma+4Dt}}$

Thackham, et al 2009, p. 223 presents the case $\sigma = D$, so sol is $u = \frac{M}{\sqrt{1+4t}}\, e^{-\frac{(x-x_0-Vt)^2}{D(1+4t)}}$

scanned

5. initial square pulse

$$(\text{IVP})\begin{cases} w_t = D w_{\xi\xi} & -\infty < \xi < \infty \\ w(\xi,0) = \begin{cases} A, & \xi_1 < \xi < \xi_2 \\ 0, & \text{otherwise} \end{cases} \\ w = 0 \text{ at } \pm\infty \end{cases}$$



Exact solution: $w(\xi,t) = \int_{-\infty}^{\infty} H(\xi - y, t)\, w_0(y)\, dy$ = Fourier-Poisson integral

$$H(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(x)^2}{4Dt}} = \text{heat kernel}$$

$$\Rightarrow \quad w(\xi,t) = A\int_{\xi_1 = y_1}^{\xi_2 = y_2} \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(\xi - y)^2}{4Dt}}\, dy = \frac{A}{\sqrt{4\pi D t}} \int_{s_1}^{s_2} e^{-s^2}\left(-\sqrt{4Dt}\right) ds = -\frac{A}{\sqrt{\pi}} \int_{s_1}^{s_2} e^{-s^2} ds$$

$$s = \frac{\xi - y}{\sqrt{4Dt}}, \quad ds = -\frac{dy}{\sqrt{4Dt}}, \qquad s_i = \frac{\xi - y_i}{\sqrt{4Dt}}$$

$$= -\frac{A}{\cancel{\sqrt{\pi}}}\,\frac{\cancel{\sqrt{\pi}}}{2}\cdot\frac{2}{\sqrt{\pi}}\left[\int_0^{s_2} - \int_0^{s_1}\right]$$

$$= +\frac{A}{2}\left[\operatorname{erf}(s_1) - \operatorname{erf}(s_2)\right] = \frac{A}{2}\left[\operatorname{erf}\left(\frac{\xi - \xi_1}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{\xi - \xi_2}{\sqrt{4Dt}}\right)\right]$$

Setting $\xi = x - Vt$, $V = \text{const}$, we obtain a sol of the advection-diffusion eqn for $u(x,t) = w(\xi,t)$:

$$(\text{IVP})\begin{cases} u_t + (Vu)_x = D u_{xx}, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = (w_0(\xi)) = \begin{cases} A, & \overset{=\xi_1}{x_1} < x < \overset{=\xi_2}{x_2} \\ 0, & \text{otherwise} \end{cases} \\ u \to 0 \text{ at } \pm\infty \end{cases}$$

$$u(x,t) = \frac{A}{2}\left[\operatorname{erf}\left(\frac{x - Vt - x_1}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{x - Vt - x_2}{\sqrt{4Dt}}\right)\right]$$