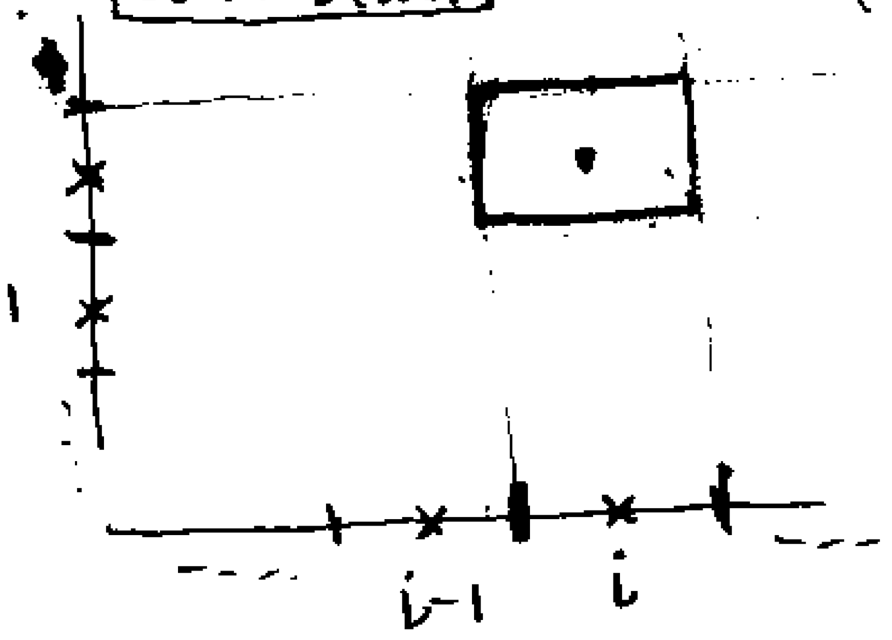
2-D case: $\frac{\partial u}{\partial t} + \nabla \cdot \vec{F} = 0$ in Ω

mesh: $\boxed{\text{Cartesian}}$ (uniform for simplicity)



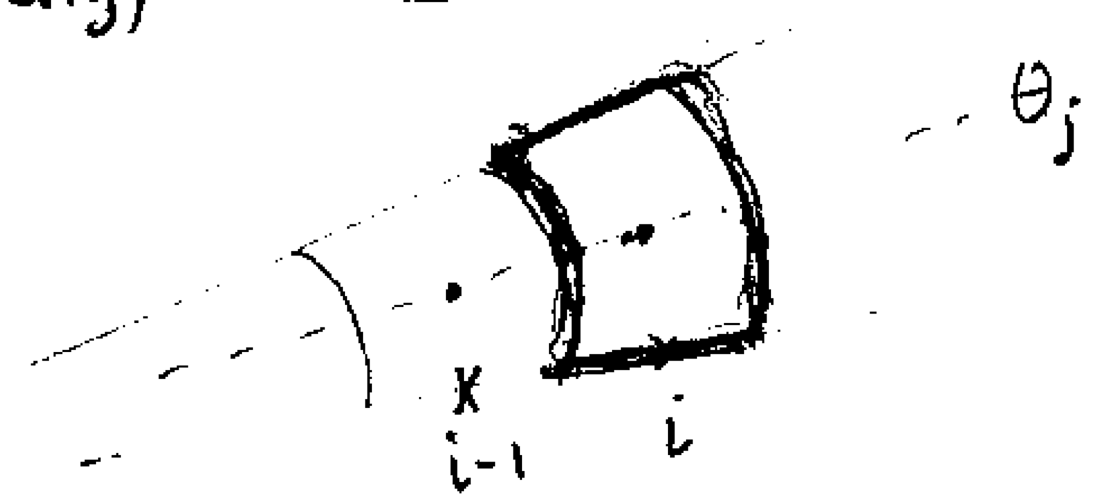
$x(0:M_x+1)$
$y(0:M_y+1)$

$V(1:M_x, 1:M_y)$ control volume ΔV_{ij}, $\Delta V_{ij} = \Delta x \Delta y$

$Ax(0:M_x+1, 1:M_y)$ x-faces: $A_{i-1/2,j} = \Delta y = A_{i+1/2,j}$

$Ay(1:M_x, 0:M_y+1)$ y-faces: $A_{i,j-1/2} = \Delta x = A_{i,j+1/2}$

$\boxed{\text{polar}}$



$r(0:M_r+1)$
$\theta(0:M_\theta+1)$

$\Delta V_{ij} = r_i \Delta r \Delta \theta$

r-faces: $A_{i-1/2,j} = r_{i-1/2}\Delta\theta$, $A_{i+1/2,j} = r_{i+1/2}\Delta\theta$

θ-faces: $A_{i,j-1/2} = \Delta r = A_{i,j+1/2}$

FV discretization: integrate over V_{ij} and over $[t_n, t_{n+1}]$:

$$\int_{t_n}^{t_{n+1}} \frac{\partial}{\partial t}\int_{V_{ij}} u \, dV dt + \int_{t_n}^{t_{n+1}}\int_{V_{ij}} \nabla\cdot\vec{F} \, dV dt = 0,$$ by Divergence Thm

$$\int_{V_{ij}} \nabla\cdot\vec{F} dV = \sum_{\text{face}} \int \vec{F}\cdot\vec{n} dA = \sum_{\text{faces}} (AF)_{\text{face}}$$

Set $U_{ij}^n = \frac{1}{\Delta V_{ij}} \int_{V_{ij}} u \, dV$ = mean u over V_{ij} at time t_n, $(AF)^{n+\theta}_{\text{face}} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (AF)_{\text{face}} dt$ = mean flow rate across face during a time step

$$\Rightarrow \Delta V_{ij}\left[U_{ij}^{n+1} - U_{ij}^n\right] + \Delta t \sum_{\text{faces}} (AF)^{n+\theta}_{\text{face}} = 0 \quad \forall i,j$$

Explicit scheme: $\theta = 0 \Rightarrow$ explicit update

$$U_{ij}^{n+1} = U_{ij}^n + \frac{\Delta t}{\Delta V_{ij}}\left[(AF)^n_{i-1/2,j} - (AF)^n_{i+1/2,j} + (AF)^n_{i,j-1/2} - (AF)^n_{i,j+1/2}\right]$$

Advection: $\vec{F} = \vec{v} u$

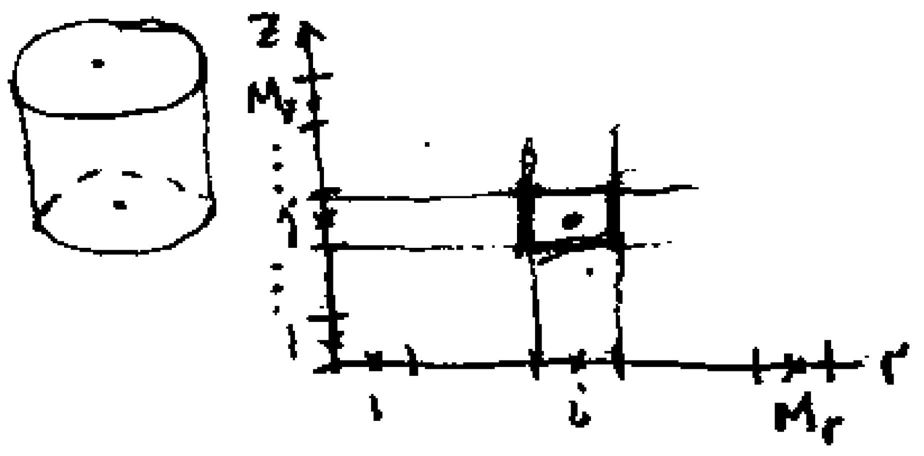
upwind fluxes:

$$F^n_{i-1/2,j} = \begin{cases} v^n_{i-1/2,j} U^n_{i-1,j} & \text{if } v>0 \\ v^n_{i-1/2,j} U^n_{ij} & \text{if } v<0 \end{cases}, \quad F^n_{i,j-1/2} = \begin{cases} v^n_{i,j-1/2} U_{i,j-1} & \text{if } v>0 \\ v^n_{i,j-1/2} U_{ij} & \text{if } v<0 \end{cases}$$

Diffusion: $\vec{F} = -D\nabla u$: $F_{i-\frac{1}{2},j} = -D\frac{U_{ij}-U_{i-1,j}}{\Delta x}$, $F_{i,j-\frac{1}{2}} = -D\frac{U_{ij}-U_{i,j-1}}{\Delta y}$

Advection-Diffusion: $\vec{F} = \vec{v}u - D\nabla u = F^{adv} + F^{dif}$

Axisymmetric in (r, z) coordinates: $u(r,z)$, independent of angle θ, $0 \le \theta \le 2\pi$



areas: $Ar_{i-\frac{1}{2},j} = 2\pi r_{i-\frac{1}{2}} \Delta z$

$$Az_{i,j-\frac{1}{2}} = \pi r_{i+\frac{1}{2}}^2 - \pi r_{i-\frac{1}{2}}^2 = 2\pi \frac{r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}}{2} \cdot \Delta r = 2\pi r_i \Delta r \quad \forall j$$

for uniform mesh
or with nodes midway of faces

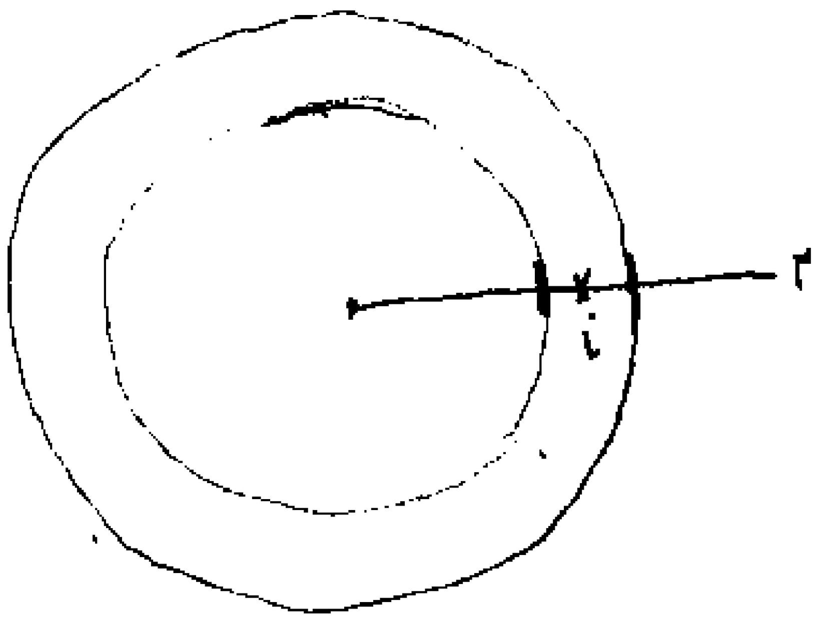
$$V_{ij} = Az_{i,j-\frac{1}{2}} \cdot \Delta z = 2\pi r_i \Delta r \cdot \Delta z$$

V_{ij}: rotated all around (a washer)

diffusion fluxes

$$Fr_{i-\frac{1}{2},j} = -D_r \frac{U_{ij} - U_{i-1,j}}{\Delta r}, \quad Fz_{i,j-\frac{1}{2}} = -D_z \frac{U_{ij} - U_{i,j-1}}{\Delta z}$$

$$U_{ij}^{n+1} = U_{ij}^n + \frac{\Delta t}{V_{ij}} \left[(AF_r)^n_{i-\frac{1}{2},j} - (AF_r)^n_{i+\frac{1}{2},j} + (AF_z)^n_{i,j-\frac{1}{2}} - (AF_z)^n_{i,j+\frac{1}{2}} \right]$$



for $u_t = \nabla \cdot (D \nabla u) = \frac{1}{r} \frac{\partial}{\partial r}\left(r D_r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial u}{\partial z} \right)$

grad, div, Laplacian in standard coordinate systems

2D Cartesian: $\nabla u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) u$ $\qquad \nabla^2 u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$

Polar: (r, θ) $\qquad \nabla u = \left(\frac{\partial u}{\partial r}, \frac{1}{r}\frac{\partial u}{\partial \theta}\right)$ $\qquad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$x = r\cos\theta$

$y = r\sin\theta$ $\qquad \nabla \cdot \vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r F_1) + \frac{\partial F_2}{\partial \theta}\right]$

Cylindrical (r, θ, z) $\qquad \nabla u = \left(\frac{\partial u}{\partial r}, \frac{1}{r}\frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial z}\right)$ $\qquad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

$x = r\cos\theta$

$y = r\sin\theta$ $\qquad \nabla \cdot \vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r F_1) + \frac{\partial F_2}{\partial \theta}\right] + \frac{\partial F_3}{\partial z}$

$z = z$

Spherical (r, φ, θ) $\qquad \nabla u = \left(\frac{\partial u}{\partial r}, \frac{1}{r}\frac{\partial u}{\partial \varphi}, \frac{1}{r\sin\varphi}\frac{\partial u}{\partial \theta}\right)$

$$\nabla \cdot \vec{F} = \frac{1}{r^2 \sin\varphi}\left(\frac{\partial}{\partial r}\left(r^2 \sin\varphi\, F_1\right) + \frac{\partial}{\partial \varphi}\left(r \sin\varphi\, F_2\right) + \frac{\partial}{\partial \theta}\left(r F_3\right)\right)$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}$$