CHAPTER I

FIRST ORDER PDEs

1. INTRODUCTION

1.A. 1st order PDE

The general 1st order PDE for the unknown $u(x, y, \dots, z)$ has the form

$$F(x, y, \dots, z,\ u_x, u_y, \dots, u_z) = 0.$$

Such equations arise in Hamilton-Jacobi theory (calculus of variations), particle mechanics, geometrical optics. We saw that conservation laws produce PDEs of the form (actually systems usually)

$$u_t + \frac{\partial}{\partial x} a(x,\, t,\, u) = b(x,\, t,\, u)$$

which are generically referred to as "conservation law form" (even if a is not the flux of u).

The theory of 1st order PDEs is essentially complete (contrary to the situation for any other general class of PDEs), due to the fact that they can be reduced to solving systems of ODEs. It is a **local theory** (just like ODE theory) and very geometrical, unlike the rest of PDE theory.

We may think of the solution $u(x, y, \dots, z)$ as a (hyper)surface, to be constructed, called an **integral surface**. We expect many solutions, a family of surfaces depending on an arbitrary function, not just on an arbitrary constant as for ODEs. How do we select an individual integral surface out of such a large family of surfaces? For an ODE, the general solution is a family of curves and we select one by requiring that it passes through a given "initial" point. Here the general solution is a family of surfaces, so naturally we can pick one by making it pass through a given "initial" curve Γ_0.

The Cauchy Problem for a 1st order PDE is the problem of finding a solution (integral) surface of the PDE which also passes through a given "initial" curve Γ_0.

1.B. Wave propagation

A **wave** is a disturbance propagating in time through a medium, carrying energy, e.g. electromagnetic waves, sound waves, water waves, seismic waves. Matter is not necessarily convected with the wave, it is the disturbance carrying energy that propagates.

Mathematical model of a wave: $u(x, t) = F(x - ct)$

undistorted wave traveling to the right with speed $c > 0$, (or to the left if $c < 0$).

What equation does $u(x, t)$ satisfy?

$$u_t + F' \cdot (-c)\ ,\ u_x = F' \quad \Rightarrow \quad u_t + cu_x = 0 \quad \text{1st order linear PDE}$$

The **linear advection equation** $u_t + cu_x = 0$ has general solution $u(x, t) = F(x - ct)$, with $F(\cdot)$ an arbitrary functions of one variable.

If we knew the initial shape of the wave: $u(x, 0) = f_0(x)\ ,\ -\infty < x < \infty$ then $F(x) = f_0(x)\ \ \forall x$ and therefore the solution of the **Initial Value Problem**, or **Cauchy Problem**

$$(CP)\begin{cases} u_t + cu_x = 0\ ,\ -\infty < x < \infty,\ t > 0 \\ u(x, 0) = f_0(x)\ ,\ -\infty < x < \infty \end{cases}$$

would be $u(x, t) = f_0(x - ct)$, provided $f_0 \subset \mathcal{C}^1(\mathbb{R})$.

Indeed, along $x - ct =$ const. we have $\frac{du}{dt}(x(t), t) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = cu_x + u_t = 0$ by the PDE.

The lines $x - ct =$ const. are called the **characteristic curves** or simply **characteristics** of the PDE $u_t + cu_x = 0$.

Note that the direction of the characteristics is $(dx, dt) = (c, 1)$ and

$$u_t + cu_x = (c, 1) \cdot (u_x, u_t) = (c, 1) \cdot \text{grad } u$$

is the **directional derivative** in this direction. So the PDE says that u does **not** change in the direction of the characteristics! This tells us how to solve the PDE:

Given $u_t + cu_x = 0$ or $(c, 1) \cdot (u_x, u_t) = 0 \Rightarrow$ characteristic direction is $(dx, dt) = (c, 1) \Rightarrow \frac{dx}{dt} = \frac{c}{1} = c \Rightarrow x = ct + \xi$, ξ = arbitrary constant, and then along such a characteristic curve: $\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = 0$ is an ODE $\Rightarrow u = k(\xi) \Rightarrow u(x, t) = k(x - ct)$, $k(\cdot)$ arbitrary. This is the **method of characteristics** by which the PDE is reduced to an ODE along each characteristic. That's how we solve 1st order PDEs.

Sinusoidal or Fourier waves

Periodic waves are of great interest and usefulness:

$$u(x, t) = A \sin(kx - wt) \quad \text{or more conveniently} \quad u(x, t) = Ae^{i(kx - wt)}$$

A = amplitude
k = wave number = number of oscillations in 2π units of space at fixed time.
w = angular frequency = number of oscillations in 2π units of time at fixed x
$p = \frac{2\pi}{w}$ = period(in time), $\frac{w}{2\pi} = \frac{1}{P} = \nu$ = frequency = number of oscillations per unit time.
$\lambda = \frac{2\pi}{k}$ = wavelength, $\frac{k}{2\pi}$ = number of oscillations per unit length

Note that $u = Ae^{ik(x - \frac{w}{k}t)}$ is a wave traveling to the right with speed $c = \frac{w}{k}$ = phase velocity = velocity of a fixed point on the wave.

2. THE CAUCHY PROBLEM FOR SEMILINEAR 1ST ORDER PDEs IN 2 VARIABLES

2.A. The Problem

We seek a solution (integral) surface $\Sigma : z = u(x, y)$ satisfying

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{1}$$

and passing through a **given curve** Γ_0, which may be represented parametrically as

$$\Gamma_0 : \quad x = x_0(s),\ y = y_0(s),\ z = u_0(s),\ s \in I = \text{interval of } \mathbb{R} \tag{2}$$

with $(x_0(s),\ y_0(s)) \in \Omega,\ s \in I$, so we want $u(x, y)$ to satisfy

$$u(x_0(s),\ y_0(s)) = z_0(s),\ s \in I. \tag{3}$$

(1),(2),(3) constitute the Cauchy Problem for the PDE (1).

At this point we don't know how smooth the data $a(x,y)$, $b(x,y)$, $c(x,y,z)$, $x_0(s)$, $y_0(s)$, $z_0(s)$ should be, so we assume them smooth enough for whatever we do to be valid! Since however u is to be $\mathcal{C}^1$ in Ω, the curve Γ_0 should be $\mathcal{C}^1$, so we assume $x_0, y_0, z_0 \in \mathcal{C}^1(I)$.

Note that (1) says $(a,\ b,\ c) \cdot (u_x,\ u_y,\ -1) = 0$, and the vector $(u_x,\ u_y,\ -1)$ is **normal** to the surface $\Sigma : z = u(x, y)$, so (1) says Σ must be, at each point, **tangent** to the "characteristic direction" (a, b, c). We try to contruct the integral surface Σ as union of curves having the characteristic direction (a, b, c) at each point, starting from the given initial curve Γ_0 which is to lie on Σ.

Pure advection with velocity $\vec{v}(\vec{x},t)$

$$u_t + \nabla\cdot\vec{F} = \overset{0}{S}$$

$$\vec{F} = \underset{\text{advective}}{u\vec{v}} + \underset{\text{non-advective}}{\vec{\mathcal{F}}}$$

Note: continuity equ must always hold: $\rho_t + \nabla\cdot(\rho\vec{v}) = 0$
Incompressible flow means $\nabla\cdot\vec{v} = 0$ $\left[\Rightarrow \frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + v\cdot\nabla\rho = 0\right]$

Pure advection means non-adv $\mathcal{F} \equiv 0$

so $u_t + \nabla\cdot(u\vec{v}) = 0$ [like Continuity equ.]

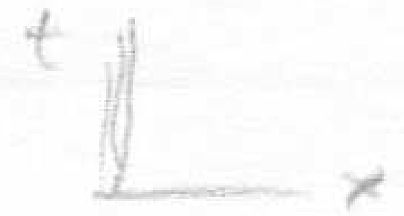
advective form: $u_t + \vec{v}\cdot\nabla u = 0$

1-D advection $\quad u_t + (uv)_x = \overset{0}{S}$ in (1-D) incompressible flow: $v_x = 0 \Rightarrow v \equiv$ const in space (uniform in space)!
call it $V \equiv$ const.

$$u_t + Vu_x = \overset{0}{S}$$
V = const.

is the simplest (1-D) linear advection equation for $u(x,t)$.

$u(x,t)$



We can view $u_t + Vu_x \equiv (u_x, u_t)\cdot(V, 1)$

$\equiv \overrightarrow{\text{grad } u}\cdot(dx, dt)$ along curves with direction $\frac{dx}{dt} = V$
called characteristics

$=$ directional deriv. of u in direction $(V, 1)$

$= 0$ so $u \equiv$ const. along characteristics!

slope $\frac{1}{V}$

$x - Vt = \xi =$ const.

This tells us how to solve the PDE $u_t + Vu_x = 0$:

Construct the characteristics: $\frac{dt}{1} = \frac{dx}{V}$ so $x = Vt + \xi$, ξ = arb. const.

On each characteristic curve $x = x(t) = Vt + \xi$:

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} \equiv u_t + Vu_x \overset{\text{PDE}}{=} 0$$

so $u(x(t),t) = \text{constant}(\xi)$, therefore $u(x,t) = \eta(\xi) = \eta(x - Vt)$, $\eta(\cdot)$ arbitrary function of one variable

Solution of IVP (Cauchy Problem): $\begin{cases} u_t + Vu_x = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) \end{cases}$

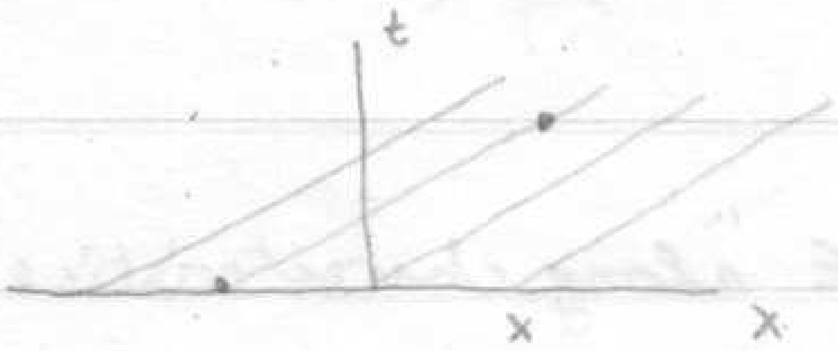
Since sol of PDE is $u(x,t) = \eta(x - Vt)$, we must have at $t=0$: $u(x,0) = \eta(x) = f(x)$

$\therefore \eta(x) \equiv f(x) \quad \therefore u(x,t) = f(x - Vt)$ = undistorted wave traveling to the right if $V > 0$
to the left if $V < 0$

Linear Advection equation

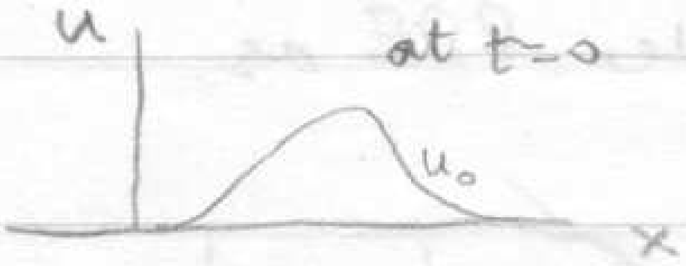
$$\begin{cases} u_t + v\,u_x = 0\,, & v = \text{const}\,, \quad -\infty < x < \infty \\ u(x,0) = u_0(x) & \text{(initial values)} \end{cases}$$

Solution: Find the characteristics: $\frac{dt}{ds} = 1 \Rightarrow s = t$; $\quad \frac{dx}{dt} = v \Rightarrow x = vt + \xi$ (straight lines)



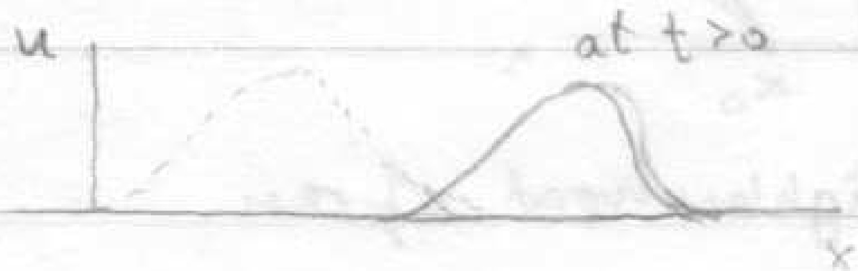
Along a characteristic $\frac{du}{dt} = 0 \Rightarrow u = k(\xi)$ (arbitrary function of ξ)

$$\Rightarrow u(x,t) = k(x - vt)$$

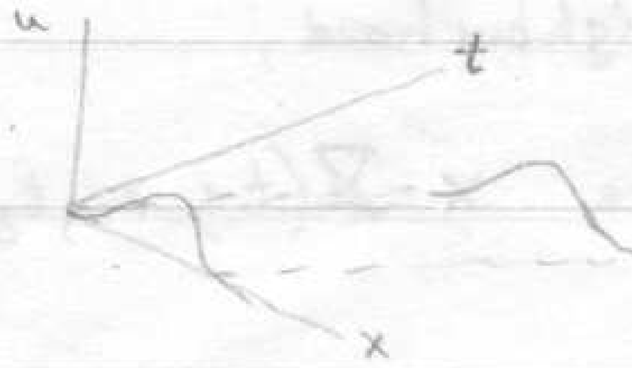


At $t = 0$: $u(x,0) = u_0(x) = k(x)$ $\forall x \Rightarrow k(x) = u_0(x)$

Therefore $u(x,t) = u_0(x - vt)$



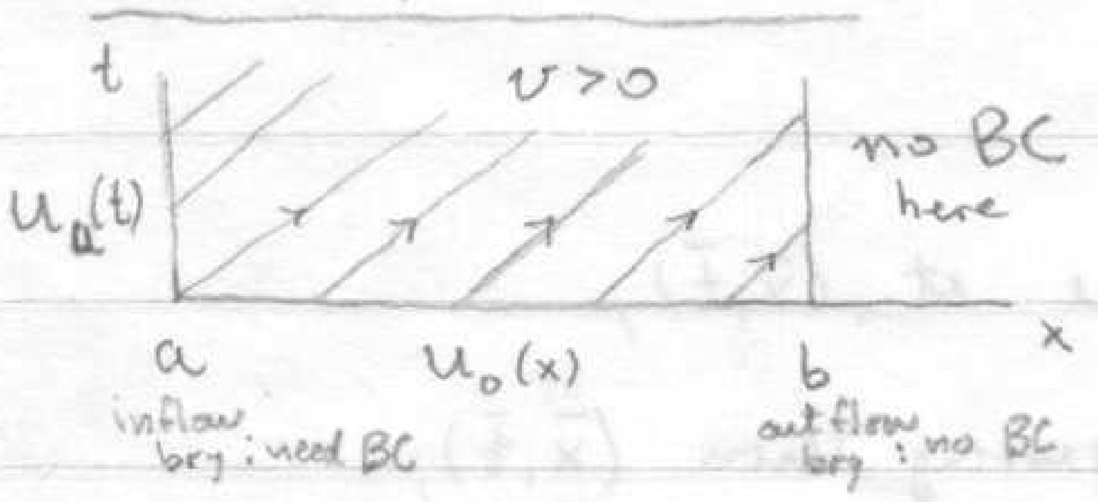
This is a traveling wave, the initial profile $u_0(x)$ moving to the right (if $v > 0$) with speed v, or to the left (if $v < 0$) " "

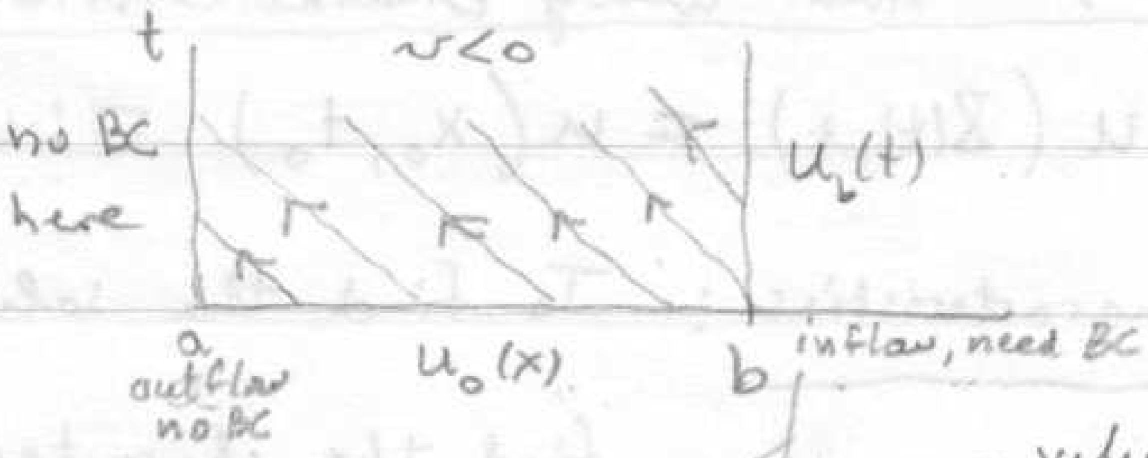


i.e. moving downstream, undistorted!

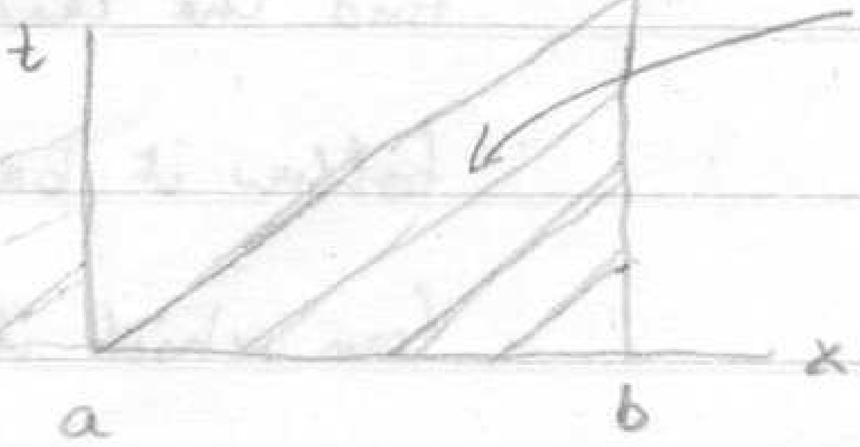
In fact, the advection equ. $u_t + v u_x = 0$ is the equation satisfied by a wave traveling undistorted with speed v. Indeed, such a wave must have the form $u(x,t) = f(x - vt)$ which satisfies: $u_t = f' \cdot (-v)$, $u_x = f' \Rightarrow u_t = u_x \cdot (-v) = -v u_x \Rightarrow u_t + v u_x = 0$.

Boundary conditions: The characteristics tell us where we need them:





Periodic BC: $u(a,t) = u(b,t)$



values depend only on $u(x,0)$

$\Rightarrow u(b,t)$ can be found

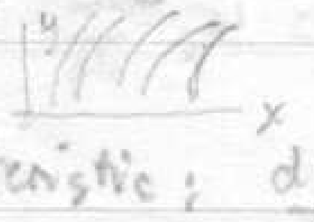
$\Rightarrow u(a,t) = u(b,t)$

so can solve in upper part

More general "quasilinear" 1st order PDE: $a u_x + b u_y = c(x,y,u)$

Characteristics: $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = ds \Rightarrow \left\{ \frac{dx}{ds} = a\,, \ \frac{dy}{ds} = b \right.$ are the characteristic curves (ODE system)

determines $C: \begin{cases} x = x(s) \\ y = y(s) \end{cases}$



Then along each characteristic: $\frac{du}{ds}(x(s), y(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = a u_x + b u_y = c$, so $\frac{du}{ds} = c$

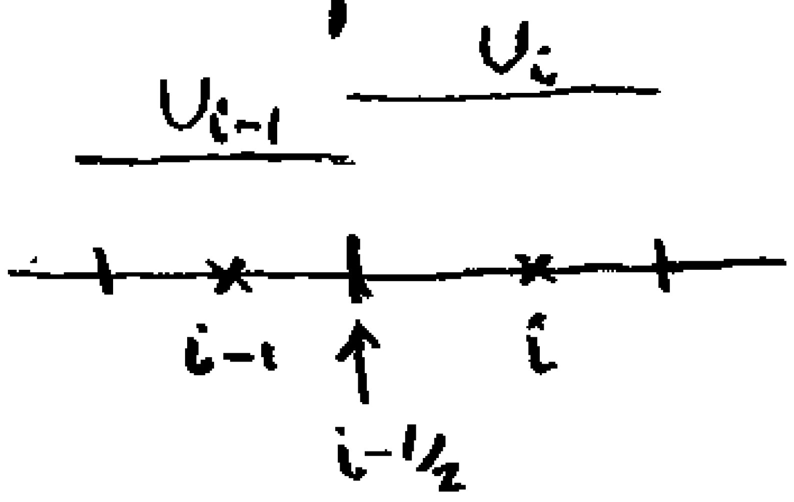
Explicit scheme for pure advection in 1-D: $F = v u$

$v(x,t) = $ velocity (known)

$\Rightarrow F^n_{i-1/2} = (v u)^n_{i-1/2}$, this requires u at face $i-1/2$, whereas the scheme updates only the mean values U_i, so we need to express face values in terms of mean values.

$$U_i^{n+1} = U_i^n + \frac{v\Delta t}{\Delta x}\left[\frac{U_{i-1}+\cancel{U_i}}{2} - \frac{\cancel{U_i}+U_{i+1}}{2}\right] = U_i^n + \frac{v\Delta t}{\Delta x}\left[U_{i-1} - U_{i+1}\right]$$

always has a negative coeff!



Simplest choice: $u_{i-1/2} \approx \dfrac{U_{i-1}+U_i}{2}$

bad idea! the resulting scheme is unstable! and unphysical: if stream flows to the right ($v>0$) then the face will only see what comes from the left (upstream), not the average of the upstream and downstream values!

The physically correct prescription is the "upwind" flux:

$$F^n_{i-1/2} = \begin{cases} v^n_{i-1/2}\, U^n_{i-1} & \text{if } v^n_{i-1/2} > 0 \\ v^n_{i-1/2}\, U^n_i & \text{if } v^n_{i-1/2} < 0 \end{cases} \qquad i = 2, \dots, M$$

The resulting explicit upwind scheme will be stable if it preserves positivity ($U_i^n > 0 \Rightarrow U_i^{n+1} > 0$) which requires all coefficients to be positive,

e.g. for $v > 0$: $U_i^{n+1} = U_i^n + \dfrac{\Delta t}{\Delta x_i}\left[v U^n_{i-1} - v U^n_i\right]$

$$= \Big[1 - \underbrace{\frac{v\Delta t}{\Delta x}}_{\mu}\Big] U_i^n + \underbrace{\frac{v\,\Delta t}{\Delta x}}_{=\mu} U^n_{i-1} \quad \text{must have } 1 - \underbrace{\frac{v\Delta t}{\Delta x}}_{\mu} \geq 0$$

$$\Rightarrow \Delta t \leq \frac{\Delta x_i}{v} \quad (\text{CFL condition}) \ \forall i$$

Similarly for $v<0$, so the CFL condition is

$$\Delta t \leqslant \frac{\min \Delta x_i}{\max|v|}$$

Note: in 1-D, $\nabla\cdot\vec{v} = 0 \Rightarrow v \equiv$ constant physical speed

Boundary condition: If $v>0$ then only the left boundary is an inflow bry, where a bry condition must be prescribed.

Dirichlet BC: $u(a,t) = u_a(t) =$ given. Set $U_0^n = u_a(t_n) =$ given, $n = 1, 2, \dots$

Then $F^n_{1/2} = v U^n_0$ and the scheme will update U^n_i, $i = 1, \dots, M$

Neumann BC: $F(a,t) =$ given. Then $F^n_{1/2} = F(a, t_n) \Rightarrow U_0 = \frac{F_{1/2}}{v}$

Similarly, if $v<0$ then BC only at $x=b$: $F^n_{M+1/2} = v\, u_b(t_n)$ or $F(b, t_n)$. $\Rightarrow U_{M+1} = \frac{F}{v}$