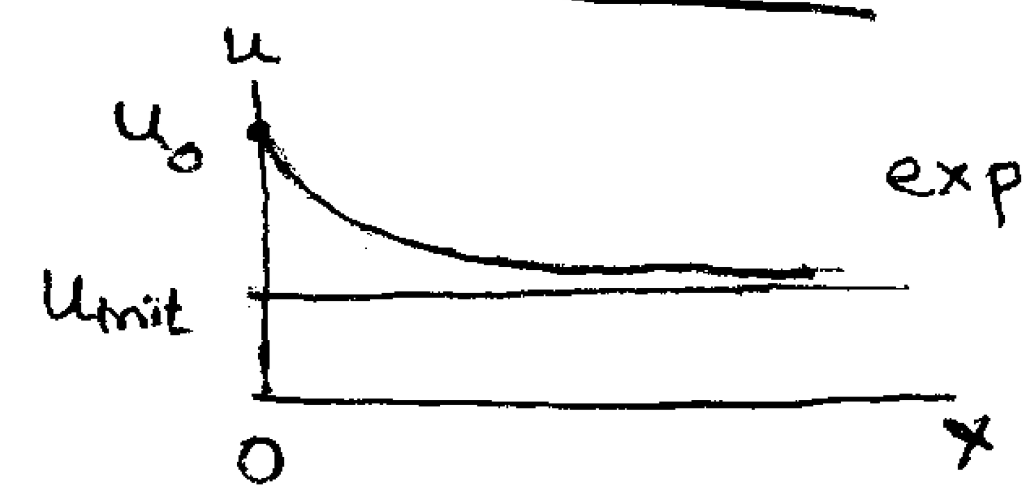
Explicitly solvable diffusion problem — for debugging 1D diffusion

The diffusion problem in semi-infinite rod, with constant data u_{init}, u_0

PDE $\quad u_t = D u_{xx}, \quad 0 < x < \infty, \quad t > 0$

IC $\quad u(x,0) = u_{init}, \quad 0 \le x < \infty$

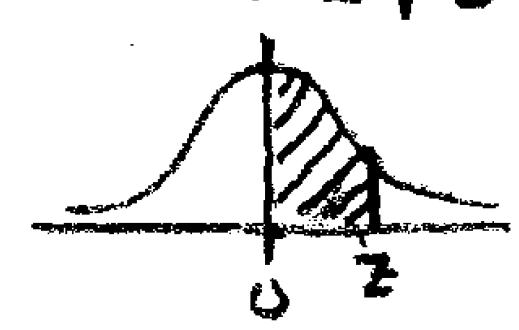
BC $\quad u(0,t) = u_0, \quad \lim_{x\to\infty} u(x,t) = u_{init}$

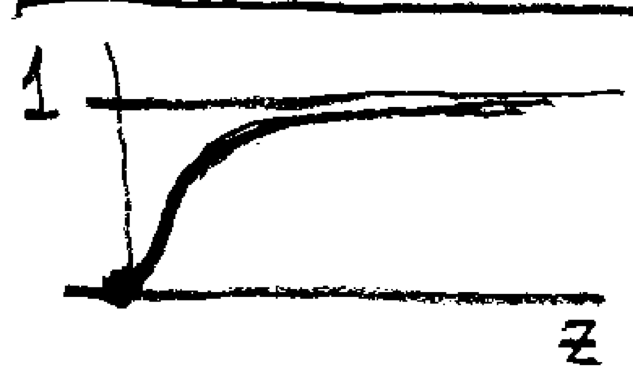


admits exact (similarity) solution: (HW1)

$$u(x,t) = u_0 + (u_{init} - u_0)\,\mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right), \quad \begin{array}{l} 0 \le x < \infty \\ t \ge 0 \end{array}$$

error function: $\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-s^2}\,ds$





$\mathrm{erf}(0) = 0, \quad \mathrm{erf}(\infty) = 1$

To test/debug diffusion code over a finite interval $[a,b]$: set $a=0$

impose exact solution at boundaries: Dirichlet BCs

$$U_0^n = u_0, \qquad U_{M+1}^n = u_b(t_n) = u_0 + (u_{init} - u_0)\,\mathrm{erf}\left(\frac{b}{2\sqrt{Dt_n}}\right)$$

time-dependent bry value

For simplicity, take $u_{init} = 0$, $u_0 = 1$

$$\Rightarrow u(x,t) = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \equiv \mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)$$

complementary error function

erf solution ... ∞ propagation speed

Important remark :

exact solution of diffusion problem with $u_{init} = 0$, $u_0 = 1$ is

$$u(x,t) = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \gneqq 0$$

So we start with $u \equiv 0$ at $t=0$, and we raise bry value to $u=1$.

Instantaneously, for any $t \gneqq 0$ we get $u(x,t) \gneqq 0$ everywhere!!!

We say the diffusion equ <u>propagates signals with infinite speed</u>!

This is a fundamental property of solutions of parabolic PDEs, in sharp contrast to hyperbolic PDEs which propagate signals with finite speed.

Also note that the initial discontinuity (at $x=0$) disappears instantly,

$u(x,t) \in C^\infty$ for $t>0$.

Thus diffusion equ is infinitely smoothing! information is lost and cannot be recovered.

As a result, the backward (in time) diffusion problem is <u>ill-posed</u>,

the past can<u>not</u> be recovered! only the future can be **determined**

again in sharp **contrast** with hyperbolic equations.

The infinite propation speed is clearly unphysical, so why is the heat equ still a good, enormously **successful** model ??? **for 200+ years**

The unphysical behavior is more "theoretical" than "practical":

erf increases to 1 extremely rapidly : **erf(6) $\approx 1 - 2 \cdot 10^{-17}$** !!!

so $u = \mathrm{erfc}$ is <u>**detectably**</u> **> 0** only for x very close to 0, (say $x < 10\sqrt{Dt}$) the rest of **the region** is essentially zero as far as any measurement can detect, in perfect agreement with experience!

$\mathrm{erfc}(5) \approx 10^{-12}$
$\mathrm{erfc}(6) \approx 2 \cdot 10^{-17}$
$\mathrm{erfc}(7) \approx 4 \cdot 10^{-23}$
$\mathrm{erfc}(10) \approx 2 \cdot 10^{-45}$