Classification of (linear) PDEs in 2 variables: $u(x,y)$

1st order linear PDEs: $a u_x + b u_y = c$ classified as hyperbolic type.
(more later ...)

2nd order linear PDEs:

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F = 0$$

classified into 3 types according to sign of discriminant:

$$B^2 - 4AC \begin{cases} > 0 & \text{hyperbolic type} \\ = 0 & \text{parabolic type} \\ < 0 & \text{elliptic type} \end{cases}$$

By changing variables, each type can be transformed into a prototype (model) PDE:

			higher dimensions (space)
hyperbolic:	wave equ:	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2 \nabla^2 u$
parabolic:	heat (diffusion) equ:	$u_t = D u_{xx}$	$u_t = D \nabla^2 u$
elliptic:	Laplace equ:	$u_{xx} + u_{yy} = 0$ (steady-state, no time-like variable)	$\nabla^2 u = 0$

IBVPs: PDE is to hold in some region $\vec{x} \in \Omega$, for some time interval $0 < t < t_{end}$ and need to also specify Initial and Boundary Conditions:

ICs for wave equ: $u(x,0) = \text{given}$, $u_t(x,0) = \text{given}$ $x \in \Omega$ (initial state)

for heat equ: $u(x,0) = \text{given}$

for Laplace equ: none, no time-like variable, describes steady-state phenomena and <u>BCs</u> along the boundary $\partial\Omega$ of the (spatial) region Ω

need <u>some</u> BC at each point of $\partial\Omega$, for all time of interest

most common and useful kinds of BCs:

I. <u>Dirichlet</u> (1^{st} kind): $u(x,t) = u_{bry}(t)$ given for $x \in \partial\Omega$, $0 < t < t_{end}$

specify bry values on $\partial\Omega$, for all time

II. <u>Neumann</u> (2^{nd} kind): $\frac{\partial u}{\partial \vec{n}}$ given for $x \in \partial\Omega$

specify bry flux on $\partial\Omega$, for all time

$\sim \nabla u \cdot \vec{n}$

III. <u>convective</u> (3^{rd} kind, Robin): $\frac{\partial u}{\partial \vec{n}} = h \cdot (u_{amb}(t) - u)$ for $x \in \partial\Omega$

bry flux proportional to $u_{amb} - u|_{\partial\Omega}$

with $u_{amb}(t)$ given

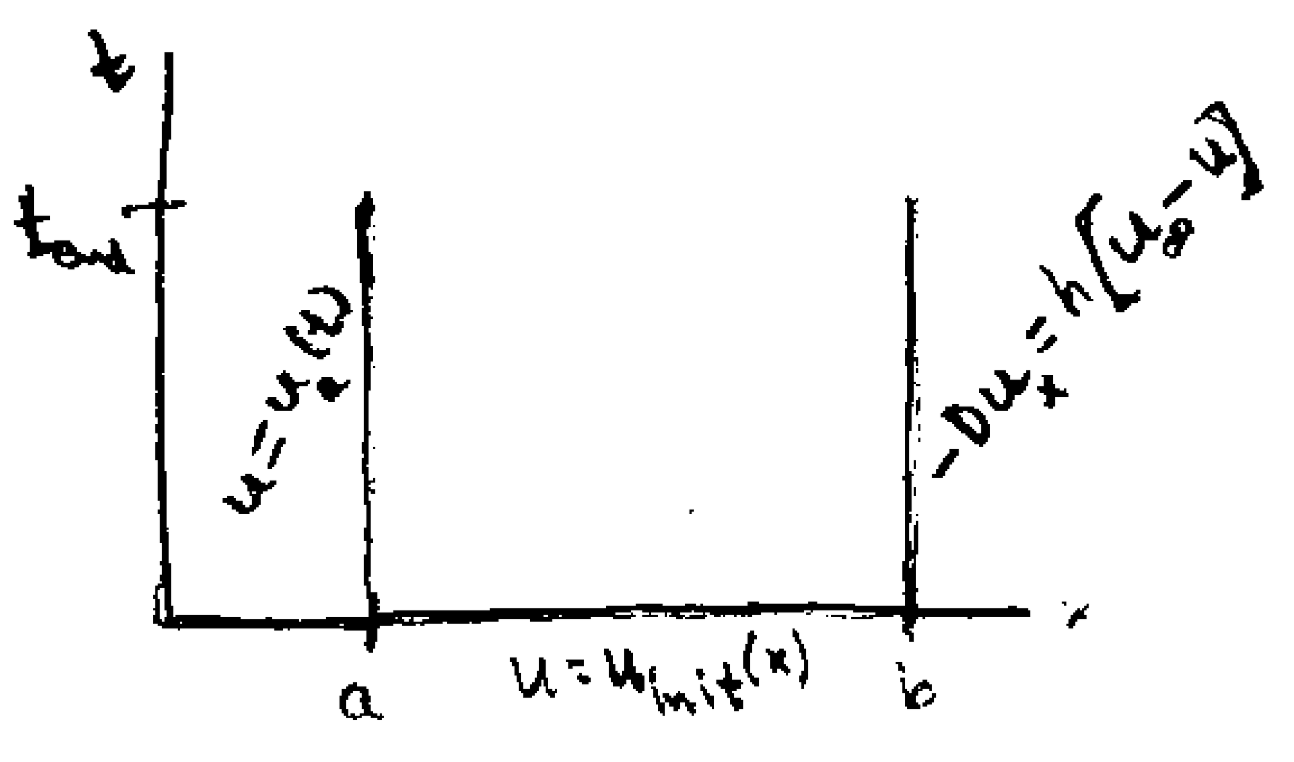
IV. periodic BCs: $u(x+P) = u(x)$, $\frac{\partial u}{\partial n}(x+P) = \frac{\partial u}{\partial n}(x)$

for some period P

V. radiation BC: $\frac{\partial u}{\partial n} = h\,(u_{amb}^4 - u^4)$

Such IBVPs are <u>well-posed problems</u>: unique solution exists that depends continuously on data

e.g. IBVP for 1-D diffusion



$u_t = D u_{xx}$, $a < x < b$, $0 < t < t_{end}$

IC: $u(x,0) = u_{init}(x)$, $a \le x \le b$

BCs: $u(a,t) = u_a(t)$, $0 < t < t_{end}$

$-Du_x(b,t) = h \cdot [u_\infty(t) - u(b,t)]$, $0 < t < t_{end}$

Behavior of solutions drastically depends on type of PDE

I. of hyperbolic (wave) equ: 1. well-posed forward and backward in time (cannot tell past or future)

2. signals propagate with finite speed

3. features (discontinuities) are carried forward in time

4. solutions can be oscillatory (no Max Principle holds)

5. ICs: $u, u_t |_{t=0}$, BCs: some BC at each pt of $\partial\Omega$ for all time $\gtrless 0$

II. of parabolic (heat) equ: 1. well-posed only forward in time (it does distinguish past from future) backward in time problem is ill-posed!

2. signals propagate with infinite speed!

3. features (discontinuities) are smoothed out instantaneously

4. Max Principle holds: max, min cannot occur inside Ω (no oscillations inside Ω)

5. IC: $u|_{t=0}$, BCs: some BC at each pt of $\partial\Omega$ forall $t>0$

III. of elliptic (Laplace) equ: 1. no time-like variable, describe steady-state phenomena

2. —

3. features are smoothed out (by "averaging" of bry values)

4. Max Principle holds: max, min cannot occur inside Ω

5. no IC, only BCs at each pt of $\partial\Omega$.

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right), \quad \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \sum_i \frac{\partial F_i}{\partial x_i}$$

$$\nabla^2 u \equiv \nabla\cdot\nabla u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \equiv \Delta u \quad \text{Laplasian operator}$$