High degree polynomial interpolation

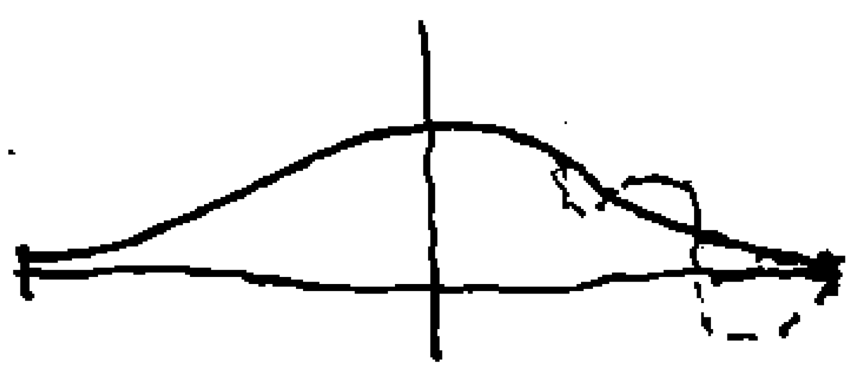
By Weierstrass Approximation Thm, any continuous function can be approximated by a polynomial uniformly.

The trouble is to construct a polynomial that also interpolates $N+1$ values of the function in interval $[a,b]$.

One would expect that given a smooth $f(x)$ we can take more and more values, closer and closer, so that $P_N \to f$ as $N \to \infty$, i.e. so that $\lim_{N\to\infty} \| f - P_N \|_\infty = 0$.

Unfortunately, this is not true!

Famous counterexample: Runge function: $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$ (it is C^∞)



If $P_N(x)$ interpolates $f(x)$ at $N+1$ equispaced nodes than $\|f - P_N\|_\infty \to \infty$ as $N \to \infty$!!!

The main trouble in this example may be traced to using equispaced nodes!

So, despite their simplicity and convenience, uniform mesh is bad in this case!

Much better are the Chebyshev nodes on $[a,b]$: $x_i = \frac{a+b}{2} + \frac{b-a}{2} \cdot \cos\left(\frac{2i-1}{K} \cdot \frac{\pi}{2}\right)$, $i = 1, \dots, K$

(at evenly spaced angles)

more nodes near the endpoints

These are zeros of Chebyshev polynomial of degree K

Polynomial interpolation behaves much better on Chebyshev nodes, for Runge function.

However, even Chebyshev nodes cannot always solve the problem: it can be shown

For any given set of nodes, there exists a continuous function f such that $\lim_{N\to\infty} \| f - P_N \|_\infty = \infty$. $\therefore$ there is no set of nodes that can work for all f!

Moral: Avoid high degree polynomial interpolants.

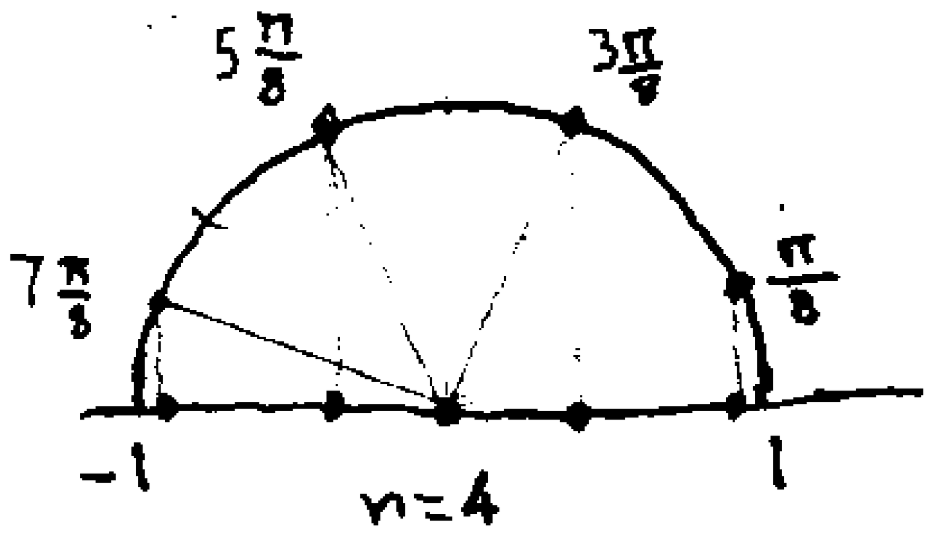
they are more and more oscillatory as N increases, avoid

Chebyshev Polynomials

$T_n(x) = \cos(n\cdot\theta)$ $n=0,1,\ldots$, $-1\le x\le 1$, $x=\cos\theta$
$= \cos(n\cdot \cos^{-1}x)$

is polynomial of deg n, with n distinct zeros in $[-1,1]$

$$t_k^{(n)} = \cos\left(\frac{2k-1}{n}\cdot\frac{\pi}{2}\right), \quad k=1,2,\ldots,n$$



First few Chebyshev polynomials:

$n=0:\quad T_0(x) = 1$

$n=1:\quad T_1(x) = \cos\theta = x$

$n=2:\quad T_2(x) = \cos(2\theta) = 2\cos^2\theta - 1 = 2x^2-1$

$n=3:\quad T_3(x) = 2x(2x^2-1) - x \equiv 4x^3-3x$

$n=4:\quad T_4(x) = \qquad = 8x^4-8x^2+1$

....

Recursion formula: $T_{n+1}(x) = 2x\cdot T_n(x) - T_{n-1}(x)$, $n=1,2,\ldots$

They have some amazing optimality properties ...

For N-th degree interpolant at $N+1$ nodes we use the zeros of $T_{N+1}(x)$
i.e. $t_{i+1}^{(N+1)}$, $i=0,1,\ldots,N$, because this choice minimizes $\max \prod_{i=0}^{N} |x-x_i|$ over $[-1,1]$

Bernstein polynomials (1912) for $f(x)$: $B_n^f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$

Thm: If $f\in C[0,1]$ then $B_n^f(x) \underset{n\to\infty}{\rightrightarrows} f(x)$ uniformly in $[0,1]$. Constructive proof of Weirstrass Approximation Thm

Used in Bezier curves, splines, ...

Bernstein basis polynomials: $\beta_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k=0,1,\ldots,n$ have nice properties (Wikipedia article)

For each n, $\{\beta_{k,n}(x)\}$ form a basis for the vector space Π_n of polynomials of degree $\le n$ (with real coefficients).

They form a partition of unity: $\sum_{k=0}^{n} \beta_{k,n}(x) = \sum \binom{n}{k} x^k (1-x)^{n-k} = (x+(1-x))^n \equiv 1$!