The Tilt Formula

Anastasiia Tsvietkova

University of Tennessee

Hyperbolic Manifolds

As a corollary of Geometrization Conjecture, many 3-manifolds have hyperbolic metric or can be decomposed into pieces with hyperbolic metric (1978, W. Thurston). In view of Mostow rigidity Th., for manifold with finite volume such metric is unique as long as it is complete. Hence, hyperbolic structure can be used to **distinguish between manifolds** (in particular, to distinguish knots).

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Hyperbolic manifold M - having a Riemannian metric of curvature -1 or, equivalently, a manifold together with the lift to its universal cover, which is hyperbolic space H^3. Covering transformations act as discrete group of fixed point free isometries Γ. The manifold M is then homeomorphic to the quotient H^3/Γ.

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A **cusp** is a neighborhood of the missing knot in the complement. It lifts to a set of horoballs with disjoint interiors in the universal cover H^n.

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Our purpose is to present a theorem which underlies SnapPea's "algorithm" for determining whether two cusped hyperbolic 3-manifolds are isometric.

Minkowski space

We work in the Minkowski space model of hyperbolic *n*-space. The **Minkowski (or Lorentzian) space** $E^{n,1}$ is the real vector space R^{n+1} with the inner product $x \cdot y = -x_0 y_0 + x_1 y_1 + \dots + x_n y_n$.

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The set of all x such that $x \cdot x = 0$ is called the **light cone** and x is called **light-like**. Vectors are **space-like** and **time-like** according to whether $x \cdot x > 0$ (outside the cone) or $x \cdot x < 0$ (inside the cone). Vector subspace is time-like iff it has a time-like vector, space-like iff every its nonzero vector is space-like.

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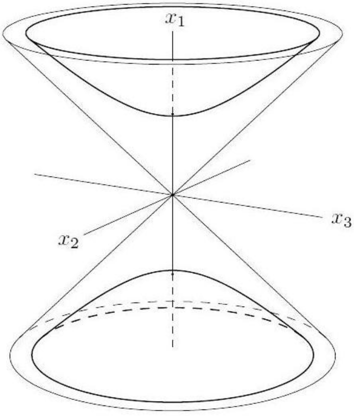
Two vectors are **orthogonal** iff their Minkowski inner product is 0.

Hyperboloid Model

A set of all x such that $x \cdot x = -1$ and $x_0 > 0$ is the **upper sheet of the hyperboloid**. It is a model for hyperbolic n-space. It asymptotically approaches, but never intersects the light-cone.



Geodesics And Hyperplanes

A **hyperbolic** *m*-**plane** is the intersection of H^n with an $(m+1)$-dimensional time-like vector subspace of $E^{n,1}$. If m=1, it is a **geodesic**.

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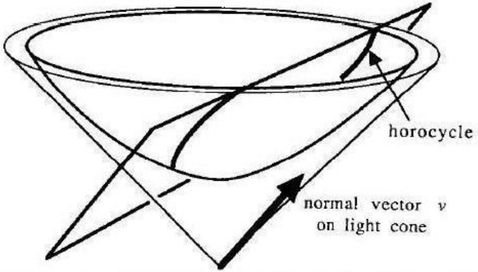
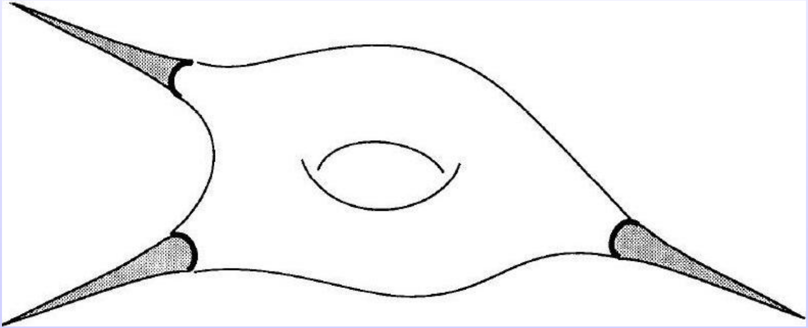


Fig. 3. A horocycle may naturally be associated to the unique vector v such that $v \cdot w = -1$ for all points w on the plane containing the horocycle. The vector v lies on the light cone, and is simultaneously parallel and orthogonal to the plane of the horocycle.

Canonical Ideal Triangulation

Consider a cusped hyperbolic *n*-manifold. Choose horocyclic cross-sections of the cusps, all bounding equal volumes. These will be the "vertices" of the triangulation. The canonical triangulation does not depend on which cross-section we choose (J. Weeks).



Definition Of The Canonical Ideal Triangulation

The preimage of the chosen cross-sections in the universal cover H^n is a set S of disjoint horocycles. Each horocycle is associated with the unique vector on the light-cone. Denote V the set of points on the light-cone, corresponding to S.

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Take a convex hull of V. The "reverse" process (project convex hull radially from the origin into H^n and then project to the original manifold) gives the canonical ideal triangulation of the hyperbolic manifold.

Canonical Ideal Triangulation

Ideal means every vertex is on the boundary, i.e. at infinity. This is true, since every point in V is on the light-cone, which is a boundary of hyperboloid. **Canonical** means that the triangulation depends only on the geometry of manifold, not on our initial choice of the vertices (i.e. of cusp cross-sections).

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SnapPea decides whether two cusped hyperbolic 3-manifolds are equivalent by computing a canonical triangulation for each. Input is an arbitrary ideal triangulation of the manifold, output - the canonical ideal triangulation. The "algorithm" relies on the **tilt formula** (by J. Weeks).

SnapPea's Algorithm

SnapPea starts with an arbitrary ideal triangulation. The cusp cross-sections define a set of points V on the light-cone as before. Instead of taking the convex hull, take hull corresponding to the faces of the initial triangulation. If such hull is convex, the corresponding triangulation is canonical. If not, we locally modify the hull to be convex.

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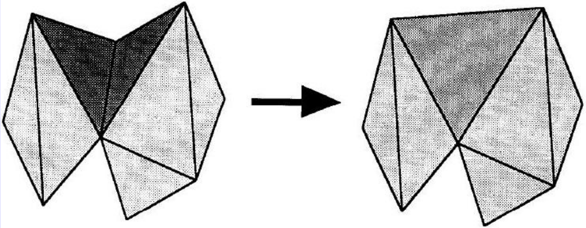

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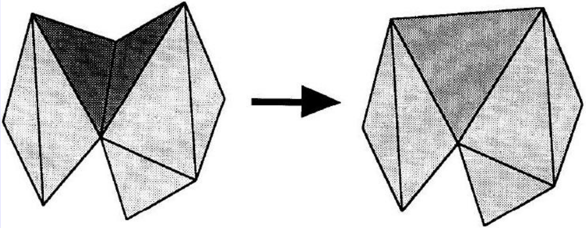

Tilt is used for recognizing the concave angles.

Definition Of The Tilt

Let T be an ideal triangulation of a cusped hyperbolic n-manifold. For an n-simplex F of T, let $\widetilde{F}$ be a lift of F to the universal cover H^n, let $\hat{F}$ be the convex hull in $E^{n,1}$ spanned by the points $\{v_0, v_1, ..., v_n\} \subset V$ corresponding to the ideal vertices of $\widetilde{F}$. The normal vector p to $\hat{F}$ is defined by $p \cdot x = -1$ for all x in $\hat{F}$.

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Let E_i be the face of F opposite the ideal vertex corresponding to v_i. Let $\hat{E}_i$ (resp. $\widetilde{E}_i$) be the face of $\hat{F}$ (resp $\widetilde{F}$) corresponding to E_i. Let m_i be the outward pointing unit normal to the hyperplane in Minkowski space containing $\hat{E}_i$ and the origin.

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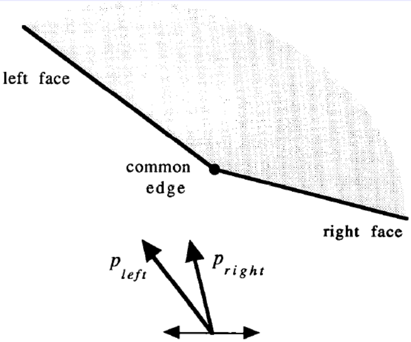
The **tilt** t_i of F relative to E_i is $m_i \cdot p$.

Tilt And Concave Angles

The dihedral angle is convex iff the normal vector to the left face p_{left} lies to the left to the normal vector to the right face p_{right}. This occurs iff the sum of tilts $t_{left} + t_{right} < 0$.



The Tilt Formula

Theorem (M. Sakuma, J. Weeks). In an ideal triangulation of a cusped hyperbolic *n*-manifold, the tilt of an ideal *n*-simplex relative to each of its codimension 1 faces may be computed as

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 1 & -\cos\theta_{01} & -\cos\theta_{02} & \cdots & -\cos\theta_{0n} \\ -\cos\theta_{10} & 1 & -\cos\theta_{12} & \cdots & -\cos\theta_{1n} \\ -\cos\theta_{20} & -\cos\theta_{21} & 1 & \cdots & -\cos\theta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n0} & -\cos\theta_{n1} & -\cos\theta_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

where t_i is the tilt relative to the face opposite vertex i, R_i is the circumradius of vertex cross-section i, and θ_{ij} is the dihedral angle between the faces opposite vertices i and j.

Proof Of The Tilt Formula: General Plan

The proof is organized in a top-down fashion. Lemma 1 shows that the vectors $\{m_0, m_1, ..., m_n\}$ form a basis for Minkowski space. Relative to this basis, $m_k = \{0, ..., 0, 1, 0, ..., 0\}$, $p = (R_0, ..., R_n)$ by Lemma 3 and the metric is given by the $(n+1) \times (n+1)$ matrix above, computed in Lemma 5.

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$$t_k = m_k \cdot p = (0, ..., 0, 1, 0, ..., 0) \cdot$$

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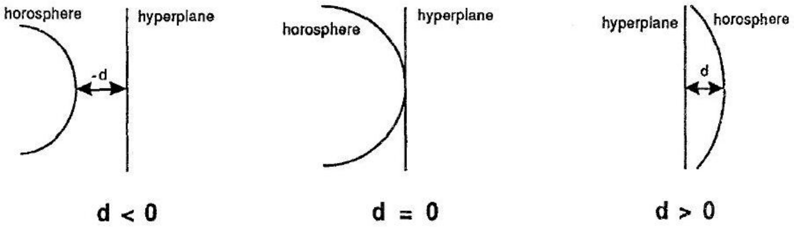
Proof Of The Tilt Formula: Lemmas

The **signed distance** from a hyperplane to horosphere is the distance by which the horosphere extends past the hyperplane.



L 1. The set $\{m_0, m_1, ..., m_n\}$ forms a basis for Minkowski space, and is dual to the basis $\{-e^{d_0} v_0, ..., -e^{d_n} v_n\}$, where d_i is a signed distance from a hyperplane of the cusp cross-section of v_i to the horosphere of the opposite face.

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Pf. For $i \neq j$, v_j is in E_i, which is orthogonal to m_i, so $m_i \cdot v_j = 0$. By Lemma 2, $m_i \cdot v_i = -e^{d_i}$, hence, the above sets are dual. The duality implies linear independence.

Proof Of The Tilt Formula: Lemmas

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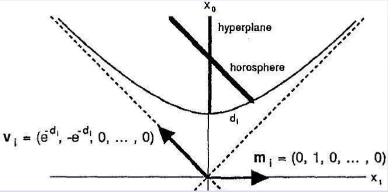
Pf. The hyperplane determined by m_i and the horosphere determined by v_i have 1 common perpendicular. Choose coordinates: the perpendicular intersects the hyperplane at $(1, 0, ..., 0)$, and the horosphere at $(coshd_i, sinhd_i, 0, ..., 0)$.



Vector v_i is on the ray $(-t, t, 0, ..., 0)$. Using $v_i \cdot w = -1$ for all w on the horosphere, get $(-t, t, 0, ..., 0) \cdot (coshd_i, sinhd_i, 0, ..., 0) = -te^{d_i} = -1$, so $t = e^{-d_i}$ and $m_i \cdot v_i = -e^{d_i}$.

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Pf. $p = \Sigma(p \cdot (-e^{d_i})v_i)\mathrm{m}_i$

$(\{m_0, m_1, ..., m_n\}$ and $\{-e^{d_0}v_0, ..., -e^{d_n}v_n\}$ are dual)

$= \Sigma e^{d_i} m_i$

$(p \cdot v_i = -1$ by the definition of $p)$

$= \Sigma R_i m_i$ (Lemma 4).

Proof Of The Tilt Formula

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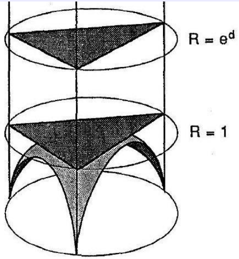
Pf. The ideal n-simplex in the upper half space model of H^n with vertex i at ∞ :



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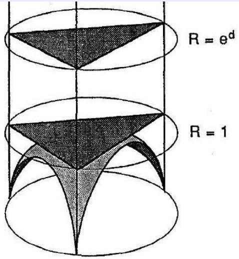

The hyperplane containing opposite face is a Euclidean hemisphere of radius 1. A vertex cross-section tangent to the opposite face has circumradius 1; at a signed distance d from the opposite face has circumradius e^d.

Proof Of The Tilt Formula

L 5. Relative to the basis $\{m_0, m_1, ..., m_n\}$, the Minkowski metric is

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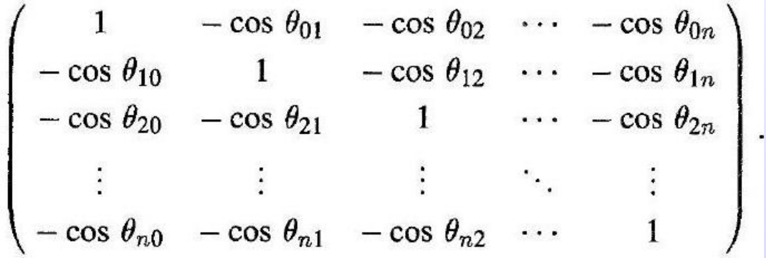

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Pf. $m's$ are unit vectors, so $m_i \cdot m_i = 1$. For $i \neq j$ the angle between m_i, m_j equals the exterior angle between E_i, E_j. By Th. of cos, $m_i \cdot m_j$ is cosine of the exterior angle or the negative of the cosine of the interior angle.

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- A database of low-complexity cusped hyperbolic 3-manifolds.
- Computation of symmetry group of hyperbolic knots and links.
- Drawing the Ford domain (dual to the canonical triangulation) for cusped hyperbolic 3-manifolds.