An alternative approach to hyperbolic structures on link complements

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Hyperbolic manifolds

As a corollary of Thurston's Hyperbolization Theorem, many **3–manifolds have hyperbolic metric** or can be decomposed into pieces with hyperbolic metric. In view of Mostow-Prasad rigidity, for a manifold with finite volume such **metric is unique** as long as it is complete.

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Hence, hyperbolic structure can be used to **distinguish between manifolds**, in particular hyperbolic knot complements. If two knot complements are distinguished, then so are the corresponding knots (Gordon-Luecke).

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Thurston demonstrated that every knot in S^3 is either a torus knot, a satellite knot, or a hyperbolic knot, and these three categories are mutually exclusive.

Why study hyperbolic links?

A knot is called **prime** if, for any decomposition as a connected sum, one of the factors is unknotted. Every knot can be uniquely decomposed as a knot sum of prime knots (Schubert). It also true for non-split links (i.e. if there is no a 2–sphere in the complement separating the link). This motivates the study of **prime knots and links**.

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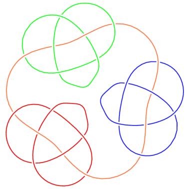

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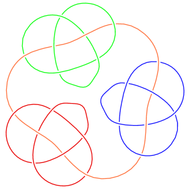
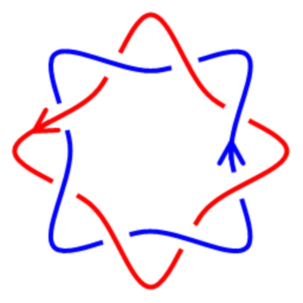


From the results of Menasco, it follows: a prime alternating diagram represents either a hyperbolic link or a torus link.

Why study hyperbolic knots and links?

The only alternating torus links are those of type $(2, n)$. From the solution of the Tait flyping conjecture (Menasco, Thistlethwaite), each reduced alternating diagram of $(2, n)$–torus link is standard.



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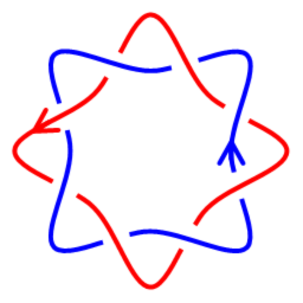

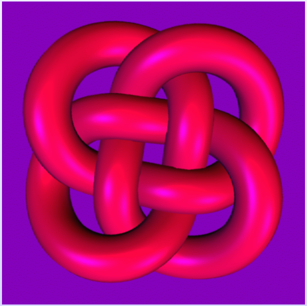

Thus one can immediately see if a prime alternating diagram represents a hyperbolic link.

Why study hyperbolic knots and links?

Hence hyperbolic geometry is an appropriate tool for the study of alternating links. The question about non-diagrammatic, i.e. topological and geometric properties that characterize alternating links, is among the fundamental questions in classical knot theory.

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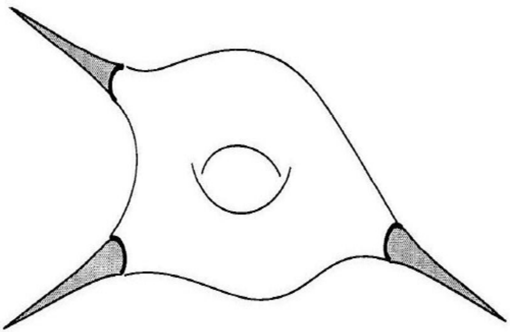
A new method for describing the structure of hyperbolic links was suggested by M. Thistlethwaite. It is applicable to all hyperbolic links, whose diagrams satisfy a few mild restrictions, but works particularly well for alternating links.

Construction

A **cusp** is a neighborhood of the missing knot in the complement.



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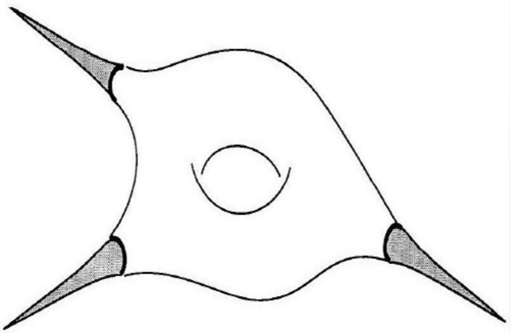
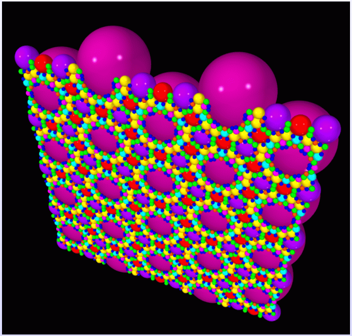
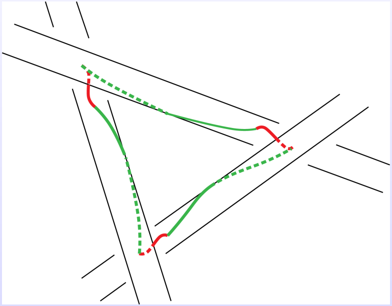




Its preimage in $\mathbb{H}^n$ is a set of horoballs. Cusps may be chosen so that the horoballs have disjoint interiors . We want to understand **horoball packings** associated to complements of hyperbolic alternating links.

Construction

The method is based on ideal polygons bounding the regions of a link diagram.



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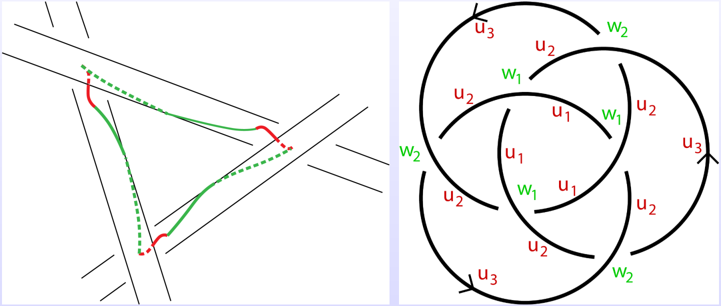

Given a diagram with n crossings of a hyperbolic link, we shall associate a complex number w_i to each of the crossings, and a complex number u_j to each of the edges.

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The **edge and crossing labels** are defined in such a way that they **contain all the geometric information** about the lifting of the link to the hyperbolic space, including all the angles and intercusp distances.

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Each region of the diagram gives rise to an identity in the labels attached to incident vertices and edges.

Hence, **directly from a link diagram** we get a system of equations, and one of its solutions gives the **hyperbolic structure**.

Computer implementation

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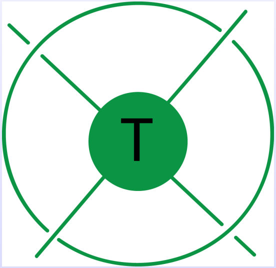
We implemented a C++ program for alternating links.

Input: a sequence of integers which decodes a 2-dimensional diagram of a link (Dowker–Thistlethwaite). It is a convenient way to store knots.

Output: edge and crossing labels.

Some applications: encircled tangles

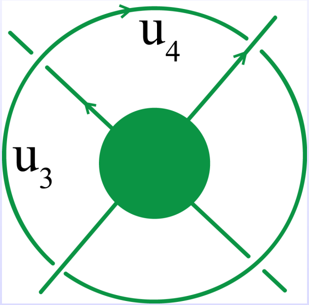
A union of an arbitrary tangle T and a simple closed curve that weaves around the ends of T in alternating fashion will be called an **encircled tangle**.



Theorem. The hyperbolic structure of the tangle T is rigid, i. e. it does not depend on the structure of a hyperbolic link containing such encircled tangle.

Some applications: tangle ratio

The **ratio** $\frac{u_4}{u_3}$ does not change, being a numerical invariant of a tangle.



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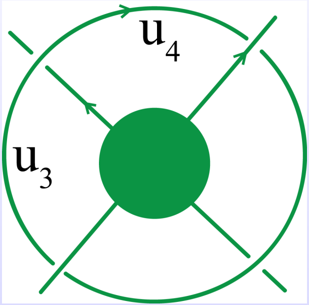
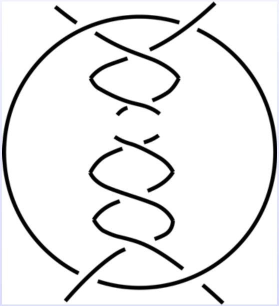
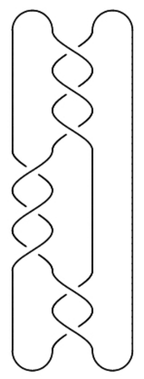


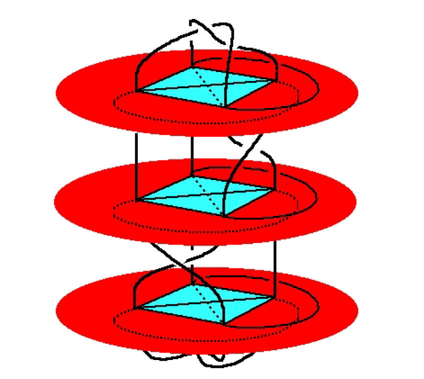


We obtained an equation for the ratio of a tangle with k twists. It can be generalized to an arbitrary rational tangle.

Some applications: volume of 2–bridged links

The formula for the **volume of 2-bridged links** was obtained using Sakuma-Weeks description of ideal triangulation together with the described method.





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Edge and crossing labels determine a representation of a **link group** into $\mathrm{PSL}_2(\mathbb{C})$. The **Wirtinger representation** of the fundamental group of a link complement can be computed directly from the labels.

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Edge and crossing labels describe the unique structure of a hyperbolic link, but depend on the diagram. To turn this information into a topological invariant, we introduced the notion of a **projection field**. It is generated over $\mathbb{Q}$ by edge and crossing labels. It contains both cusp and trace fields.