

# ASYMPTOTIC RAYS

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## Abstract

We prove that a graph  $\Gamma$  is asymptotically isomorphic to the ray if and only if  $\Gamma$  is uniformly spherically bounded and is of bounded local degrees. This problem arose in combinatorics and was posed in [3] (Problem 10.1). **AMS Subject Classification:** 54A05, 54E15, 05C60  
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A ray  $\mathcal{R}$  is a non-oriented graph with the set of vertices  $\omega = \{0, 1, \dots\}$  and the set of edges  $\{(i, i+1) : i \in \omega\}$ . An asymptotic ray is a non-oriented graph, asymptotically isomorphic to the ray. The notion of asymptotic isomorphism arose from the following general combinatoric scheme.

A *ball structure* is a triple  $\mathcal{B} = (X, P, B)$  where  $X, P$  are non-empty sets, and for all  $x \in X$  and  $\alpha \in P$ ,  $B(x, \alpha)$  is a subset of  $X$  which is called a ball of radius  $\alpha$  around  $x$ . It is supposed that  $x \in B(x, \alpha)$  for all  $x \in X, \alpha \in P$ . The set  $X$  is called the *support* of  $\mathcal{B}$ ,  $P$  is called the *set of radiuses*.

Given any  $x \in X, A \subseteq X, \alpha \in P$ , we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure  $\mathcal{B} = (X, P, B)$  is called

- *lower symmetric* if,  $\forall \alpha, \beta \in P \exists \alpha', \beta' \in P$  such that  $\forall x \in X$

$$B^*(x, \alpha') \subseteq B(x, \alpha), \quad B(x, \beta') \subseteq B^*(x, \beta);$$

- *upper symmetric* if  $\forall \alpha, \beta \in P \exists \alpha', \beta' \in P$  such that  $\forall x \in X$

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if  $\forall \alpha, \beta \in P \exists \gamma \in P$  such that  $\forall x \in X$

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$

- *upper multiplicative* if  $\forall \alpha, \beta \in P \exists \gamma \in P$  such that  $\forall x \in X$

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

Let  $\mathcal{B} = (X, P, B)$  be a lower symmetric and lower multiplicative ball structure. Then the family

$$\left\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \right\}$$

is a base of entourages for some (uniquely determined) uniformity on  $X$ . For information about uniformity and uniform topological spaces see [1]. On the other hand, if  $U \subseteq X \times X$  is a uniformity on  $X$ , then the ball structure  $(X, U, B)$  is lower symmetric and lower multiplicative, where  $B(x, U) = \{y \in X : (x, y) \in U\}$ . Thus, the lower symmetric and lower multiplicative ball structures can be naturally identified with uniform topological spaces.

A ball structure which is upper symmetric and upper multiplicative is called *balleans*. The balleans appeared independently in asymptotic topology [2] under name coarse structures and in combinatorics [3]. Directly from the definition it follows that the balleans can be considered as asymptotic counterparts of uniform topological spaces. For more details about this duality see [2, 3]. The role of morphisms in the category of uniform topological spaces is played by uniformly continuous mappings, and that is why it is necessary to define its asymptotic equivalents.

Let  $\mathcal{B}_1 = (X_1, P_1, B_1)$  and  $\mathcal{B}_2 = (X_2, P_2, B_2)$  be balleans.

A mapping  $f : X_1 \rightarrow X_2$  is called a *<-mapping* if  $\forall \alpha \in P_1 \exists \beta \in P_2$  such that:

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

A bijection  $f : X_1 \rightarrow X_2$  is called an *asymptotic isomorphism* (briefly *asymorphism*) between  $\mathcal{B}_1$  and  $\mathcal{B}_2$  if  $f$  and  $f^{-1}$  are  $\prec$ -mappings, and  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are called *asymorphic*.

For an arbitrary ballean  $\mathcal{B} = (X, P, B)$  a family  $\mathfrak{S}$  of subsets of the support  $X$  is called *uniformly bounded*, if  $\exists \alpha \in P$  such that  $\forall F \in \mathfrak{S}, F \subseteq B(x, \alpha)$  for some  $x \in X$ . A bijection  $f : X_1 \rightarrow X_2$  is an *asymorphism* between  $\mathcal{B}_1$  and  $\mathcal{B}_2$  iff for any uniformly bounded family  $\mathfrak{S}$  of subsets of  $X_1$ , the family  $f(\mathfrak{S}) = \{f(F) : F \in \mathfrak{S}\}$  is uniformly bounded in  $\mathcal{B}_2$ , and for any uniformly bounded family  $\mathfrak{S}'$  of subsets of  $X_2$ , the family  $f^{-1}(\mathfrak{S}') = \{f^{-1}(F) : F \in \mathfrak{S}'\}$  is uniformly bounded in  $\mathcal{B}_1$ .

Every metric space  $(X, d)$  determines the ballean  $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$ , where  $\mathbb{R}^+$  is the set of non-negative real numbers,

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

A ballean  $\mathcal{B}$  is called *metrizable* if  $\mathcal{B}$  is asymorphic to  $\mathcal{B}(X, d)$  for some metric space  $(X, d)$ . A criterion of metrizability of a ballean can be found in [3] (Theorem 9.1).

Every connected graph  $\Gamma(V, E)$  with the set of vertices  $V$  and the set of edges  $E$  determines the metric space  $(V, d)$ , where  $d(u, v)$  is the length of the shortest path from  $u$  to  $v$ . Thus, for every graph  $\Gamma$  there is a ballean  $\mathcal{B}(\Gamma) = \mathcal{B}(V, d)$  with the support  $V$ , which corresponds to  $\Gamma$ . A ballean  $\mathcal{B}$  is called a *graph ballean* if  $\mathcal{B}$  is asymorphic to the ballean  $\mathcal{B}(\Gamma)$  of some connected graph  $\Gamma$ . A criterion of the graph ballean can be found in [3] (Theorem 9.2). In what follows we will consider only connected graphs.

The following lemma makes clear the notion of  $\prec$ -mapping for graph ballians.

**Lemma1.** *Let  $\Gamma_1(V_1, E_1)$  and  $\Gamma_2(V_2, E_2)$  be connected graphs. Then the following statements are equivalent:*

- (i)  $f$  is a  $\prec$ -mapping between  $\mathcal{B}(\Gamma_1)$  and  $\mathcal{B}(\Gamma_2)$ ;
- (ii) there exists a natural number  $m$ , such that  $f(B_1(v, 1)) \subseteq B_2(f(v), m)$  for every vertex  $v \in V_1$ , where  $B_1$  and  $B_2$  are balls of corresponding radiuses in  $\Gamma_1$  and  $\Gamma_2$ ;
- (iii) there exists a natural number  $m$ , such that  $d_2(f(v), f(u)) \leq md_1(v, u)$  for any  $v, u \in V_1$ , where  $d_1$  and  $d_2$  are metrics on  $V_1$  and  $V_2$ .

*Proof.* (i)  $\Rightarrow$  (ii) follows directly from the definition of  $\prec$ -mapping.

(ii)  $\Rightarrow$  (iii) If  $d(v, u) = 1$ , then  $u \in B_1(v, 1)$  so  $d_2(f(v), f(u)) \leq m$ . For any  $v, u \in V_1$  we choose the shortest path  $v = v_0, v_1, \dots, v_k = u$  between  $u$  and  $v$ . Since  $d_2(f(v_i), f(v_{i+1})) \leq m$  for every  $i \in \{0, \dots, k-1\}$ , we have  $d_2(f(v), f(u)) \leq m \cdot k = md_1(v, u)$ .

(iii)  $\Rightarrow$  (i). It suffices to notice that (iii) is equivalent to:  $f(B_1(v, k)) \subseteq B_2(f(v), m \cdot k)$  for every  $v \in V_1$ .  $\square$

Thus,  $\prec$ -mappings of graph ballians is the Lipschitz mappings between the metric spaces of the corresponding graphs.

A graph  $\Gamma(V, E)$  is called *bounded* if there exists a natural number  $m$  such that  $d(u, v) \leq m$  for all  $u, v \in V$ . If  $\Gamma_1(V_1, E_1)$ ,  $\Gamma_2(V_2, E_2)$  are graphs and  $\Gamma_1$

is bounded, then  $\mathcal{B}(\Gamma_1)$  is asyomorphic with  $\mathcal{B}(\Gamma_2)$  iff  $\Gamma_2$  is bounded and  $|V_1| = |V_2|$ . Hence, the problem of asyomorphism between graph balleans concerns unbounded graphs only.

The simplest example of unbounded graph is a ray - a non-oriented graph  $\mathcal{R}$  with a set the vertices  $\omega = \{0, 1, \dots\}$  and the set of edges  $\{(i, i + 1) : i \in \omega\}$ . We say that a graph  $\Gamma$  is an *asyptotic ray* if the balleans  $\mathcal{B}(\Gamma)$  and  $\mathcal{B}(\mathcal{R})$  are asyomorphic. Remind that the degree  $\rho(v)$  of a vertex  $v$  of a graph  $\Gamma$  is the number of edges incident to  $v$ .

**Lemma2.** *Let  $\Gamma(V, E)$  be an asyptotic ray. Then there exists a natural number  $m$ , such that  $\rho(v) \leq m$  for every  $v \in V$ .*

*Proof.* . Fix a bijection  $f : V \rightarrow \omega$ , which is a  $\prec$ -mapping between  $\mathcal{B}(\Gamma)$  and  $\mathcal{B}(\mathcal{R})$ . Choose a natural number  $k$  such that  $f(B(v, 1)) \subseteq B'(f(v), k)$  for every  $v \in V$ , where  $B$  and  $B'$  are balls of corresponding radiuses in  $\Gamma$  and  $\mathcal{R}$ . It is obvious that  $|B'(u, k)| \leq 2k + 1$  for all  $u \in \omega$ . Since  $f$  is a bijection then  $|B(v, 1)| \leq 2k + 1$  for all  $v \in V$ , so we can put  $m = 2k$ .  $\square$

Let  $\Gamma(V, E)$  be an arbitrary graph. For any  $v \in V$  and  $k \in \omega$ , we put  $S(v, k) = \{u \in V : d(u, v) = k\}$ . An injective sequence of vertices  $(v_n)_{n \in \omega}$  of  $\Gamma$  is called *an arrow*, which starts from the vertex  $v_0$ , if  $(v_i, v_{i+1}) \in E$  and  $v_i \in S(v_0, i)$  for all  $i \in \omega$ . A graph  $\Gamma$  is called locally finite if the degrees of all its vertices are finite. By Kønig lemma, from each vertex of an infinite locally finite graph starts at least one arrow. In view of this remark and Lemma2, the following theorem gives the characterization of asyptotic rays.

**Theorem 1.** *Let  $\Gamma(V, E)$  be an infinite graph,  $s$  be a natural number such that  $\rho(v) \leq s$  for all  $v \in V$ , and let  $(a_n)_{n \in \omega}$  be an arrow in  $\Gamma$ ,  $A = \{a_n : n \in \omega\}$ . Then the following statements are equivalent:*

- (i)  $\Gamma$  is an asyptotic ray;
- (ii) there exists a natural number  $r$  such that  $V = B(A, r)$ ;
- (iii) the family  $\{S(a_0, n) : n \in \omega\}$  of subsets of  $V$  is uniformly bounded in  $B(\Gamma)$ .

*Proof.* (i)  $\Rightarrow$  (ii). Fix an asyomorphism  $f : V \rightarrow \omega$  between  $\mathcal{B}(\Gamma)$  and  $\mathcal{B}(\mathcal{R})$ . Using Lemma1, we choose a natural number  $m$  such that  $|f(u) - f(v)| \leq m$  for all  $(u, v) \in E$ . Consider an injective sequence  $(f(a_n))_{n \in \omega}$  in  $\omega$  and note that  $|f(a_{n+1}) - f(a_n)| \leq m, n \in \omega$ . Put  $k = \max\{m, \min_{n \in \omega} f(a_n)\}$  and note that every segment  $[i, i + k], i \in \omega$  contains at least one element of the sequence  $(f(a_n))_{n \in \omega}$ . Since  $f^{-1} : \omega \rightarrow V$  is a  $\prec$ -mapping, there exists a natural number  $r$  such that  $f^{-1}([i, i + k]) \subseteq B(f^{-1}(i), r)$  for all  $i \in \omega$ . Since  $\omega = \bigcup_{i \in \omega} [i, i + k]$ ,  $f$  is a bijection and every segment  $[i, i + k]$  contains at least one element of the sequence  $(f(a_n))_{n \in \omega}$ , then  $V = \bigcup B(a_n, r) = B(A, r)$ .

(ii)  $\Rightarrow$  (i). Fix an arbitrary number  $n \in \omega$  and note that  $S(a_0, n) \cap B(a_k, r) = \emptyset$ , if  $|k - n| > r$ . Since  $V = B(A, r)$ , then  $S(a_0, n) \subseteq$

$\bigcup\{B(a_k, r) : |k - n| \leq r\} \subseteq B(a_0, 2r)$  and the family  $\{S(a_0, n) : n \in \omega\}$  is uniformly bounded.

(iii)  $\Rightarrow$  (i). Define a bijection  $f$  between  $V$  and  $\omega$  in a such way: put  $f(a_0) = 0$ , and then numerate in an arbitrary order the elements of  $S(a_0, 1)$ , then the elements of  $S(a_0, 2)$  and so on. It follows clearly from the uniform boundedness of the family  $\{S(a_0, n) : n \in \omega\}$  and  $V = \bigcup_{n \in \omega} S(a_0, n)$ , that  $f$  is an isomorphism.  $\square$

In conclusion we precise Theorem1 for trees. Let  $(a_n)_{n \in \omega}$  be an arrow in some tree  $T$ . After deletion of edges (but not vertices) of the arrow, the tree  $T$  disintegrates into trees  $T(a_n)$  with the roots  $a_n, n \in \omega$ .

**Theorem 2.** *Let  $T(V, E)$  be an infinite tree,  $s$  be a natural number such that  $\rho(v) \leq s$ , for all  $v \in V$ , and let  $(a_n)_{n \in \omega}$  be an arrow in  $T$ ,  $A = \{a_n : n \in \omega\}$ . The tree  $T$  is an asymptotic ray iff there exists a natural number  $t$ , such that  $|V(T(a_n))| \leq t$  for all  $n \in \omega$ , where  $V(T(a_n))$  is the set of vertices of  $T(a_n)$ .*

*Proof.* Let  $T$  be an asymptotic ray. Using Theorem1, choose a natural number  $r$  such that  $V = B(A, r)$ . Since  $T$  is a tree,  $T(a_n) \subseteq B(a_n, r)$  for all  $n \in \omega$ . Since  $\rho(v) \leq s$  for all  $v \in V$ , then  $|B(a_n, r)| \leq s^r + 1$ . If  $|V(T(a_n))| \leq t$  for all  $n \in \omega$ , then  $T(a_n) \subseteq B(a_n, t)$ ,  $V = B(A, t)$  and we can use Theorem1.  $\square$

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