

Strong instability of standing waves for nonlinear Klein-Gordon equation and Klein-Gordon-Zakharov system

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Abstract

The orbital instability of ground state standing waves $e^{i\omega t}\phi_\omega(x)$ for the nonlinear Klein-Gordon equation has been known in the domain of all frequencies ω for the supercritical case and for frequencies strictly less than a critical frequency ω_c in the subcritical case. We prove the strong instability of ground state standing waves for the entire domain above. For the case when the frequency is equal to the critical frequency ω_c we prove strong instability for all radially symmetric standing waves $e^{i\omega_c t}\varphi(x)$. We prove similar strong instability results for the Klein-Gordon-Zakharov system.

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1 Introduction and Main Results

We study the strong instability of standing wave solutions $e^{i\omega t}\varphi(x)$ for the nonlinear Klein-Gordon equation of the form

$$(1.1) \quad \partial_t^2 u - \Delta u + u = |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 2$, $1 < p < 1 + 4/(N - 2)$, $-1 < \omega < 1$, and $\varphi \in H^1(\mathbb{R}^N)$ is a nontrivial solution of

$$(1.2) \quad -\Delta \varphi + (1 - \omega^2)\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N.$$

We also study the same problem for the Klein-Gordon-Zakharov system

$$(1.3) \quad \partial_t^2 u - \Delta u + u + nu = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

$$(1.4) \quad c_0^{-2}\partial_t^2 n - \Delta n = \Delta(|u|^2), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N = 2, 3$, $c_0 > 0$ is a constant. The system (1.3)-(1.4) describes the interaction of Langmuir wave and ion acoustic wave in a plasma. The complex valued function u denotes the fast time scale component of electric field raised by electrons, and the real valued function n denotes the deviation of ion density (see [34, 4, 8]).

From the result of Genible and Velo [10], the Cauchy problem for (1.1) is locally well-posed in the energy space $X := H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Thus, for any $(u_0, u_1) \in X$ there exists a unique solution $\vec{u} := (u, \partial_t u) \in C([0, T_{\max}); X)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ such that either $T_{\max} = \infty$ (global existence) or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|\vec{u}(t)\|_X = \infty$ (finite time blowup). Moreover, the solution $u(t)$ satisfies the conservation laws of energy and charge:

$$E(\vec{u}(t)) = E(u_0, u_1), \quad Q(\vec{u}(t)) = Q(u_0, u_1), \quad t \in [0, T_{\max}),$$

where

$$(1.5) \quad E(u, v) = \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

$$(1.6) \quad Q(u, v) = \operatorname{Im} \int_{\mathbb{R}^N} \bar{u}v \, dx.$$

Let $\phi_\omega \in H^1(\mathbb{R}^N)$ be the ground state (the least energy solution) of (1.2). We refer to [2, 30] for the existence of ϕ_ω , and to [13] for the uniqueness of ϕ_ω . The stability of standing waves $e^{i\omega t}\phi_\omega$ for (1.1) has been studied by many authors. First, we consider the orbital stability of $e^{i\omega t}\phi_\omega$. Shatah [27] proves that $e^{i\omega t}\phi_\omega$ is orbitally stable if $p < 1 + 4/N$ and $\omega_c < |\omega| < 1$, where

$$(1.7) \quad \omega_c = \sqrt{\frac{p-1}{4-(N-1)(p-1)}}.$$

Shatah and Strauss [29] prove that $e^{i\omega t}\phi_\omega$ is orbitally unstable when $p < 1 + 4/N$ and $|\omega| < \omega_c$ or when $p \geq 1 + 4/N$ and $|\omega| < 1$. Here, we say that a standing wave solution $e^{i\omega t}\varphi$ is orbitally stable for (1.1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, u_1) \in X$ satisfies $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \delta$, then the solution $u(t)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ exists globally and satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|\vec{u}(t) - e^{i\theta}(\varphi(\cdot + y), i\omega\varphi(\cdot + y))\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\varphi$ is said to be orbitally unstable.

Next, we consider instability of $e^{i\omega t}\phi_\omega$ in a stronger sense. Berestycki and Cazenave [1] prove that the ground state standing wave $e^{i\omega t}\phi_\omega$ for (1.1) is very strongly unstable (see Definition 1 below) when the frequency $\omega = 0$ (see also [26]). Shatah [28] proves that the ground state standing wave $e^{i\omega t}\phi_\omega$ for nonlinear Klein-Gordon equations with general nonlinearity is strongly unstable (see Definition 2 below) when $\omega = 0$ and $N \geq 3$. Recently, the authors [23] prove that the ground state standing wave $e^{i\omega t}\phi_\omega$ for (1.1) is very strongly unstable when $|\omega| \leq \sqrt{(p-1)/(p+3)}$ and $N \geq 3$. Here, we give the definitions of very strong instability and strong instability.

Definition 1 (very strong instability) We say that $e^{i\omega t}\varphi$ is *very strongly unstable* for (1.1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$ such that $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ blows up in finite time.

Definition 2 (strong instability) We say that $e^{i\omega t}\varphi$ is *strongly unstable* for (1.1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$ such that $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ either blows up in finite time or exists globally and satisfies $\limsup_{t \rightarrow \infty} \|\vec{u}(t)\|_X = \infty$.

Note that, by the definitions, if $e^{i\omega t}\varphi$ is very strongly unstable then it is strongly unstable, and that if $e^{i\omega t}\varphi$ is strongly unstable then it is orbitally unstable.

Before stating our main results, we recall instability results for the nonlinear Schrödinger equation

$$(1.8) \quad i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Let $\omega > 0$ and $\phi_\omega \in H^1(\mathbb{R}^N)$ be the ground state of

$$(1.9) \quad -\Delta\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N.$$

It is known that for any $\omega > 0$ the standing wave solution $e^{i\omega t}\phi_\omega$ for (1.8) is orbitally stable when $1 < p < 1 + 4/N$, and it is very strongly unstable when $1 + 4/N < p < 1 + 4/(N-2)$ (see [1, 7]). Moreover, for the critical

case $p = 1 + 4/N$, for any $\omega > 0$ and any nontrivial solution $\varphi \in H^1(\mathbb{R}^N)$ of (1.9), it is known that the standing wave $e^{i\omega t}\varphi$ is very strongly unstable for (1.8) (see [32]). For general theory of orbital stability and instability of solitary waves, we refer to Grillakis, Shatah and Strauss [11, 12].

We state our main results.

Theorem 1 *Let $N \geq 2$, $1 < p < 1 + 4/(N - 2)$, $\omega \in (-1, 1)$ and ϕ_ω be the ground state of (1.2). Assume that $|\omega| \leq \omega_c$ if $p < 1 + 4/N$, where the critical frequency ω_c is given by (1.7). Then, the standing wave $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equation (1.1) is strongly unstable in the sense of Definition 2.*

Can we refine further this instability result? Namely, can we prove in certain cases that standing wave $e^{i\omega t}\phi_\omega$ for (1.1) is very strongly unstable in the sense of Definition 1? The result of Cazenave [5] gives an answer of this question for the restricted range for the exponent p of nonlinearity $1 < p \leq 5$ for $N = 2$ and $1 < p \leq N/(N - 2)$ for $N \geq 3$. Cazenave proves that any global solution $u(t)$ of (1.1) is uniformly bounded in X , i.e., $\sup_{t \geq 0} \|\bar{u}(t)\|_X < \infty$, if $1 < p \leq 5$ and $N = 2$, and if $1 < p \leq N/(N - 2)$ and $N \geq 3$. Therefore, for this range of the exponent p , Theorem 1 together with the result of Cazenave gives us a very strongly instability result in the sense of Definition 1 for ground state standing waves $e^{i\omega t}\phi_\omega$ of (1.1). Using an argument in Merle and Zaag [18], we can extend the result of Cazenave and prove the uniform boundedness of global solutions of (1.1) in X when $1 < p < 1 + 4/(N - 1)$ and $N \geq 2$. The following Lemma holds.

Lemma 2 *Let $N \geq 2$ and $1 < p < 1 + 4/(N - 1)$. If $\bar{u} \in C([0, \infty), X)$ is a global solution of (1.1), then $\sup_{t \geq 0} \|\bar{u}(t)\|_X < \infty$.*

Therefore, from Theorem 1 and Lemma 2, we deduce the following.

Corollary 3 *In addition to the assumptions in Theorem 1, let $1 < p \leq 1 + 4/(N - 1)$ if $N = 2, 3$, and that $1 < p < 1 + 4/(N - 1)$ if $N \geq 4$. Then, the ground state standing wave $e^{i\omega t}\phi_\omega$ for (1.1) is very strongly unstable in the sense of Definition 1.*

Remark. Let us mention that when the exponent p of nonlinearity is in the range $1 + 4/(N - 1) < p < 1 + 4/(N - 2)$ we were unable to give better instability results than those in Theorem 1 for ground state standing waves $e^{i\omega t}\phi_\omega$ of (1.1) for large frequencies $|\omega| > \sqrt{(p - 1)/(p + 3)}$. The very strong instability result for small frequencies $|\omega| \leq \sqrt{(p - 1)/(p + 3)}$ and $N \geq 3$ is given in [23]. The following theorem is an important contribution of Kenji

Nakanishi on the very strong instability in this area for large p and large frequencies ω .

Theorem A (due to Kenji Nakanishi) *Let $N \geq 2$, $1 + 4/N \leq p < 1 + 4/(N - 2)$, $|\omega| < 1$ and ϕ_ω be the ground state of (1.2). Then, the standing wave $e^{i\omega t}\phi_\omega$ for the nonlinear Klein-Gordon equation (1.1) is very strongly unstable in the sense of Definition 1.*

This way, we have the entire picture for the very strong instability of ground state standing waves.

For the critical frequency $\omega = \omega_c$ in the case $1 < p < 1 + 4/N$, we can prove a much more general instability result for standing waves which are not necessarily related to the ground state.

Theorem 4 *Let $N \geq 2$, $1 < p < 1 + 4/N$ and $\varphi \in H^1(\mathbb{R}^N)$ be any nontrivial, radially symmetric solution of (1.2) with $\omega = \omega_c$. Then, the standing wave solution $e^{i\omega_c t}\varphi$ of (1.1) is very strongly unstable in the sense of Definition 1. The same assertion is true for $\omega = -\omega_c$.*

For the existence of infinitely many radially symmetric solutions of (1.2), we refer to [3]. As mentioned above, a similar result of Theorem 4 is known for the nonlinear Schrödinger equation (1.8) in the critical case $p = 1 + 4/N$ without assuming the radial symmetry of solution of (1.9) and the restriction on space dimensions $N \geq 2$ (see [32]).

The proofs of Theorems 1 and 4 are based on using local versions of the virial type identities. To prove strong instability of the ground state for the case $\omega = 0$ and $N \geq 3$, Shatah in [28] considers a local version of the following identity

$$(1.10) \quad \frac{d}{dt} \operatorname{Re} \int_{\mathbb{R}^N} x \cdot \nabla u \partial_t \bar{u} \, dx = NK_1(\bar{u}(t)),$$

$$K_1(u, v) := -\frac{1}{2}\|v\|_2^2 + \left(\frac{1}{2} - \frac{1}{N}\right) \|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Since the integral in the left-hand side of (1.10) is not well-defined on the energy space X , one needs to approximate the weight function x in (1.10) by suitable bounded functions. To control error terms by the approximation, initial perturbations are restricted to being radially symmetric and the decay estimate for radially symmetric functions in $H^1(\mathbb{R}^N)$:

$$(1.11) \quad \|w\|_{L^\infty(|x| \geq m)} \leq Cm^{-(N-1)/2} \|w\|_{H^1}$$

(see [30]) is employed. The assumption $N \geq 2$ is needed here. In the case $N = 1$, we expect similar very strong instability results for the standing

waves. This kind of approach has been also used for blowup problems of the nonlinear Schrödinger equation (1.8) (see, e.g., [21, 22, 15, 16, 17, 19, 20]).

In the proof of Theorem 1 for the case $p \geq 1 + 4/N$, we use a local version of the virial identity

$$(1.12) \quad -\frac{d}{dt} \operatorname{Re} \int_{\mathbb{R}^N} \{2x \cdot \nabla u + Nu\} \partial_t \bar{u} \, dx = P(u(t)),$$

where

$$(1.13) \quad P(u) := 2\|\nabla u\|_2^2 - \frac{N(p-1)}{p+1} \|u\|_{p+1}^{p+1}.$$

Namely, instead of the left hand side of (1.12), which is not well defined in the energy space X , we use (2.6) with conveniently chosen weights (see the beginning of Section 2).

Note that (1.12) follows from (1.10) and

$$(1.14) \quad \begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 &= \frac{d}{dt} \operatorname{Re} \int_{\mathbb{R}^N} u \partial_t \bar{u} \, dx = -K_2(\bar{u}(t)), \\ K_2(u, v) &= -\|v\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 - \|u\|_{p+1}^{p+1}, \end{aligned}$$

and that the functional P appears in the virial identity for the nonlinear Schrödinger equation (1.8):

$$(1.15) \quad \frac{d^2}{dt^2} \|xu(t)\|_2^2 = 4P(u(t)).$$

The case $p < 1 + 4/N$ is more delicate. Here we use a local version of the identity

$$(1.16) \quad -\frac{d}{dt} \operatorname{Re} \int_{\mathbb{R}^N} \{2x \cdot \nabla u + (N + \alpha)u\} \partial_t \bar{u} \, dx = K(\bar{u}(t)),$$

where $\alpha := 4/(p-1) - N$ and

$$(1.17) \quad K(u, v) := -\alpha\|v\|_2^2 + \alpha\|u\|_2^2 + (\alpha + 2)\{\|\nabla u\|_2^2 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}\}$$

(cf. [29, page 185]). Note that

$$(1.18) \quad \begin{aligned} K(u, v) &= P(u) + \alpha K_2(u, v) \\ &= -2(\alpha + 1)\|v - i\omega u\|_2^2 + 2(\alpha + 2)(E - \omega Q)(u, v) \\ &\quad - 2\alpha\omega Q(u, v) - 2\{1 - (\alpha + 1)\omega^2\}\|u\|_2^2, \end{aligned}$$

and that $1 - (\alpha + 1)\omega^2 > 0$ if $|\omega| > \omega_c$, and correspondingly $1 - (\alpha + 1)\omega^2 = 0$ if $|\omega| = \omega_c$. Again instead of the left hand side of (1.16) we use (2.7) with conveniently chosen weights.

Next, we consider the Klein-Gordon-Zakharov system (1.3)-(1.4). The well-posedness of the Cauchy problem for (1.3)-(1.4) in the energy space is studied by Ozawa, Tsutaya and Tsutsumi [25]. Here, the energy space Y is defined by $Y = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times \dot{H}^{-1}(\mathbb{R}^N)$. When $N = 3$ and $c_0 \neq 1$, it is proved in [25] that for any $(u_0, u_1, n_0, n_1) \in Y$ there exists a unique solution $\mathbf{u} := (u, \partial_t u, n, \partial_t n) \in C([0, T_{\max}); Y)$ of (1.3)-(1.4) with initial data $\mathbf{u}(0) = (u_0, u_1, n_0, n_1)$ satisfying the conservation laws of the energy $H(\mathbf{u}(t)) = H(\mathbf{u}(0))$ and the charge $Q(\mathbf{u}(t)) = Q(\mathbf{u}(0))$ for all $t \in [0, T_{\max})$, where Q is defined by (1.6) and

$$(1.19) \quad H(u, v, n, \nu) = \frac{1}{2}\|v\|_2^2 + \frac{1}{4c_0^2}\|\nu\|_{\dot{H}^{-1}}^2 \\ + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\|n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 n \, dx.$$

The case $N = 3$ and $c_0 = 1$ is treated in [24, 31], where the global small data solutions result is presented. For the case $N = 2$ by using the idea of the paper of Ozawa, Tsutaya and Tsutsumi [25] we can prove the local well posedness of the Klein-Gordon-Zakharov system (1.3)-(1.4) in the energy space Y for all $c_0 > 0$.

We study instability of standing wave solutions

$$(u_\omega(t, x), n_\omega(t, x)) = (e^{i\omega t}\phi_\omega(x), -|\phi_\omega(x)|^2)$$

for (1.3)-(1.4), where $-1 < \omega < 1$, $N = 2, 3$, and $\phi_\omega \in H^1(\mathbb{R}^N)$ is the ground state of

$$(1.20) \quad -\Delta\varphi + (1 - \omega^2)\varphi - |\varphi|^2\varphi = 0, \quad x \in \mathbb{R}^N.$$

By a similar method as in the proof of Theorem 1 for the case $p \geq 1 + 4/N$ together with an argument in Merle [17] for the Zakharov system, we have the following.

Theorem 5 *Let $N = 2, 3$, $\omega \in (-1, 1)$, ϕ_ω be the ground state of (1.20), and $c_0 \neq 1$ if $N = 3$. Then, the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ of KGZ system (1.3)-(1.4) is strongly unstable in the following sense. For any $\lambda > 1$, the solution $\mathbf{u}(t)$ of (1.3)-(1.4) with initial data $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ either blows up in finite time or exists globally and satisfies $\limsup_{t \rightarrow \infty} \|\mathbf{u}(t)\|_Y = \infty$.*

Remark. It is known (see [4, Theorem 3]) that the negative initial energy $H(\mathbf{u}(0))$ implies that the solution $\mathbf{u}(t)$ of (1.3)-(1.4) either blows up in finite time or blows up in infinite time, namely the solution exists globally and satisfies the asymptotic condition $\limsup_{t \rightarrow \infty} \|\mathbf{u}(t)\|_Y = \infty$. Since the energy

$$H(\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0) > 0$$

for λ close to 1, the result in [4] is not applicable to Theorem 5.

Next, we consider the very strong instability of $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ for (1.3)-(1.4). Since the second equation (1.4) of the KGZ system is massless, it seems difficult to obtain the uniform boundedness of global solutions for (1.3)-(1.4) similar to Lemma 2. Therefore, for the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ we do not deduce a very strong instability similar to the instability result in Corollary 3 of Theorem 1. However, using the method in our previous paper [23], we obtain the following very strong instability result for small frequencies.

Theorem 6 *Let $N = 3$, $c_0 \neq 1$, $|\omega| < 1/\sqrt{3}$ and ϕ_ω be the ground state of (1.20). Then, the standing wave $(e^{i\omega t}\phi_\omega, -|\phi_\omega|^2)$ of the KGZ system (1.3)-(1.4) is very strongly unstable in the following sense. For any $\lambda > 1$, the solution $\mathbf{u}(t)$ of (1.3)-(1.4) with the initial data $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ blows up in a finite time.*

Remark. In Theorem 6, the case $\omega = 0$ is proved by Gan and Zhang [9].

The plan of this paper is as follows. In Section 2, we prove Theorems 1 and 4 and Lemma 2 for the nonlinear Klein-Gordon equation (1.1). The proof of Theorem A is given at the end of Section 2. Section 3 is devoted to applications to the Klein-Gordon-Zakharov system (1.3)-(1.4), and we prove Theorems 5 and 6.

2 Proof of Theorems for NLKG

In this section, we first prove Theorems 1 and 4.

We start with a convenient choice of the weight functions, as follows. Let $\Phi \in C^2([0, \infty))$ be a non-negative function such that

$$\Phi(r) = \begin{cases} N & \text{for } 0 \leq r \leq 1, \\ 0 & \text{for } r \geq 2, \end{cases} \quad \Phi'(r) \leq 0 \text{ for } 1 \leq r \leq 2.$$

For $m > 0$, we put

$$(2.1) \quad \Phi_m(r) = \Phi\left(\frac{r}{m}\right), \quad \Psi_m(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1} \Phi_m(s) ds.$$

Then, Φ_m and Ψ_m satisfy the following properties.

Lemma 7 For $m > 0$, we have

$$(2.2) \quad \Phi_m(r) = N, \quad \Psi_m(r) = r, \quad 0 \leq r \leq m,$$

$$(2.3) \quad \Psi'_m(r) + \frac{N-1}{r} \Psi_m(r) = \Phi_m(r), \quad r \geq 0,$$

$$(2.4) \quad |\Phi_m^{(k)}(r)| \leq \frac{C}{m^k}, \quad r \geq 0, \quad k = 0, 1, 2,$$

$$(2.5) \quad \Psi'_m(r) \leq 1, \quad r \geq 0.$$

Proof. Properties (2.2)-(2.4) follow from the definition (2.1). We show (2.5). Integrating by part implies

$$Nr^{N-1}\Psi_m(r) = \int_0^r Ns^{N-1}\Phi_m(s) ds = r^N\Phi_m(r) - \int_0^r s^N\Phi'_m(s) ds.$$

Thus, by (2.3), we have

$$\Psi'_m(r) = \Phi_m(r) - \frac{N-1}{r} \Psi_m(r) = \frac{1}{N} \Phi_m(r) + \frac{N-1}{Nr^N} \int_0^r s^N \Phi'_m(s) ds.$$

Since $\Phi_m(r) \leq N$ and $\Phi'_m(r) \leq 0$ for $r \geq 0$, we have (2.5). \square

Lemma 8 Let $u(t)$ be a radially symmetric solution of (1.1), and put

$$(2.6) \quad I_m^1(t) = 2 \operatorname{Re} \int_{\mathbb{R}^N} \Psi_m \partial_r u \partial_t \bar{u} dx + \operatorname{Re} \int_{\mathbb{R}^N} \Phi_m u \partial_t \bar{u} dx,$$

$$(2.7) \quad I_m^2(t) = I_m^1(t) + \alpha \operatorname{Re} \int_{\mathbb{R}^N} u \partial_t \bar{u} dx,$$

where $\alpha := 4/(p-1) - N$. Then, there exists a constant $C_0 > 0$ independent of m such that

$$(2.8) \quad -\frac{d}{dt} I_m^1(t) \leq P(u(t)) + \frac{N(p-1)}{p+1} \int_{|x| \geq m} |u(t, x)|^{p+1} dx + \frac{C_0}{m^2} \|u(t)\|_2^2,$$

$$(2.9) \quad -\frac{d}{dt} I_m^2(t) \leq K(\bar{u}(t)) + \frac{N(p-1)}{p+1} \int_{|x| \geq m} |u(t, x)|^{p+1} dx + \frac{C_0}{m^2} \|u(t)\|_2^2$$

for all $t \in [0, T_{\max})$.

Proof. We multiply the equation (1.1) by $\Psi_m \overline{\partial_r u}$ and by $\Phi_m \bar{u}$ respectively, and have

$$\begin{aligned} -\frac{d}{dt} 2 \operatorname{Re} \int_{\mathbb{R}^N} \Psi_m \partial_r u \partial_t \bar{u} \, dx &= \int_{\mathbb{R}^N} \left(\Psi'_m + \frac{N-1}{r} \Psi_m \right) |\partial_t u|^2 \, dx \\ &+ \int_{\mathbb{R}^N} \left(\Psi'_m - \frac{N-1}{r} \Psi_m \right) |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \left(\Psi'_m + \frac{N-1}{r} \Psi_m \right) |u|^2 \, dx \\ &+ \frac{2}{p+1} \int_{\mathbb{R}^N} \left(\Psi'_m + \frac{N-1}{r} \Psi_m \right) |u|^{p+1} \, dx, \end{aligned}$$

and

$$\begin{aligned} -\frac{d}{dt} \operatorname{Re} \int_{\mathbb{R}^N} \Phi_m u \partial_t \bar{u} \, dx &= - \int_{\mathbb{R}^N} \Phi_m |\partial_t u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \Delta \Phi_m |u|^2 \, dx \\ &+ \int_{\mathbb{R}^N} \Phi_m |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \Phi_m |u|^2 \, dx - \int_{\mathbb{R}^N} \Phi_m |u|^{p+1} \, dx. \end{aligned}$$

By (2.3) in Lemma 7, we have the identity

$$-\frac{d}{dt} I_m^1(t) = 2 \int_{\mathbb{R}^N} \Psi'_m |\nabla u|^2 \, dx - \frac{p-1}{p+1} \int_{\mathbb{R}^N} \Phi_m |u|^{p+1} \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \Delta \Phi_m |u|^2 \, dx.$$

The inequality (2.8) follows from Lemma 7. Finally, (2.9) follows from (2.8), (1.14) and (1.18). \square

First, we consider the case $p \geq 1 + 4/N$. We define the functional

$$(2.10) \quad J_\omega(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1-\omega^2}{2} \|u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1},$$

and consider the following constrained minimization problem

$$(2.11) \quad d_\omega^1 = \inf \{ J_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}, P(u) = 0 \}$$

and the set

$$(2.12) \quad \mathcal{R}_\omega^1 = \{ (u, v) \in X : (E - \omega Q)(u, v) < d_\omega^1, P(u) < 0 \},$$

where E and Q are the energy and the charge respectively, and the functional P is defined by (1.13).

Note that

$$(2.13) \quad (E - \omega Q)(u, v) = J_\omega(u) + \frac{1}{2} \|v - i\omega u\|_2^2,$$

$$(2.14) \quad P(u) = 2 \partial_\lambda J_\omega(\lambda^{N/2} u(\lambda))|_{\lambda=1}.$$

Lemma 9 *Let $N \geq 2$, $1 + 4/N \leq p < 1 + 4/(N - 2)$ and $\omega \in (-1, 1)$. Then, we have the following.*

- (i) $J_\omega(u) - \frac{1}{N(p-1)}P(u) > d_\omega^1$ for all $u \in H^1(\mathbb{R}^N)$ satisfying $P(u) < 0$.
- (ii) The minimization problem (2.11) is attained at the ground state ϕ_ω of (1.2).
- (iii) $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^1$ for all $\lambda > 1$.

Proof. (i) We put

$$(2.15) \quad \begin{aligned} J_\omega^1(u) &:= J_\omega(u) - \frac{1}{N(p-1)}P(u) \\ &= \left\{ \frac{1}{2} - \frac{2}{N(p-1)} \right\} \|\nabla u\|_2^2 + \frac{1-\omega^2}{2} \|u\|_2^2. \end{aligned}$$

Note that $1/2 - 2/N(p-1) \geq 0$ by the assumption $p \geq 1 + 4/N$. Let $u \in H^1(\mathbb{R}^N)$ satisfy $P(u) < 0$. Then, we have $u \neq 0$, and there exists $\lambda_1 \in (0, 1)$ such that $P(\lambda_1 u) = 0$. By (2.11), we have $d_\omega^1 \leq J_\omega(\lambda_1 u) = J_\omega^1(\lambda_1 u) < J_\omega^1(u)$.

(ii) For the case $p > 1 + 4/N$, see [6, Proposition 8.2.4], and for $p = 1 + 4/N$, see [19, Proposition 2.5].

(iii) By (2.13), we have

$$\begin{aligned} (E - \omega Q)(\lambda(\phi_\omega, i\omega\phi_\omega)) &= J_\omega(\lambda\phi_\omega) \\ &= \lambda^2 \left(\frac{1}{2} \|\nabla \phi_\omega\|_2^2 + \frac{1-\omega^2}{2} \|\phi_\omega\|_2^2 \right) - \frac{\lambda^{p+1}}{p+1} \|\phi_\omega\|_{p+1}^{p+1}. \end{aligned}$$

Since $J_\omega(\phi_\omega) = d_\omega^1$, $\partial_\lambda J_\omega(\lambda\phi_\omega)|_{\lambda=1} = 0$ and $\partial_\lambda^2 J_\omega(\lambda\phi_\omega)|_{\lambda=1} < 0$, we have $(E - \omega Q)(\lambda(\phi_\omega, i\omega\phi_\omega)) < d_\omega^1$ for all $\lambda > 1$. Similarly, we have $P(\lambda\phi_\omega) < 0$ for all $\lambda > 1$. Hence, we have $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^1$ for all $\lambda > 1$. \square

Lemma 10 *Suppose that $N \geq 2$, $1 + 4/N \leq p < 1 + 4/(N - 2)$ and $\omega \in (-1, 1)$. If $(u_0, u_1) \in \mathcal{R}_\omega^1$, then the solution $u(t)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ satisfies*

$$(2.16) \quad -\frac{1}{N(p-1)}P(u(t)) > d_\omega^1 - (E - \omega Q)(u_0, u_1), \quad t \in [0, T_{\max}).$$

Proof. First, we show that $P(u(t)) < 0$ for all $t \in [0, T_{\max})$. Suppose that there exists $t_1 \in (0, T_{\max})$ such that $P(u(t_1)) = 0$ and $P(u(t)) < 0$ for $t \in [0, t_1)$. Then, by Lemma 9 (i) and (2.15), we have

$$\left\{ \frac{1}{2} - \frac{2}{N(p-1)} \right\} \|\nabla u\|_2^2 + \frac{1-\omega^2}{2} \|u(t)\|_2^2 > d_\omega^1 > 0, \quad t \in [0, t_1).$$

Thus, we have $u(t_1) \neq 0$. Therefore, by (2.11), we have $d_\omega^1 \leq J_\omega(u(t_1))$. While, since $(u_0, u_1) \in \mathcal{R}_\omega^1$, E and Q are conserved, and by (2.13), we have $J_\omega(u(t_1)) \leq (E - \omega Q)(\vec{u}(t_1)) < d_\omega^1$. This is a contradiction. Hence, we have $P(u(t)) < 0$ for all $t \in [0, T_{\max})$. From this fact, Lemma 9 (i) and (2.13), we obtain (2.16). \square

Proof of Theorem 1 for the case $p \geq 1 + 4/N$. Let $\lambda > 1$ be fixed and denote

$$\delta := \frac{N(p-1)}{2} \{d_\omega^1 - (E - \omega Q)(\lambda(\phi_\omega, i\omega\phi_\omega))\}.$$

Then, by Lemma 9 (iii), we have $\delta > 0$. Suppose that the solution $u(t)$ of (1.1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ exists for all $t \in [0, \infty)$ and is uniformly bounded in X , i.e.,

$$(2.17) \quad M_1 := \sup_{t \geq 0} \|\vec{u}(t)\|_X < \infty.$$

Since $u(t)$ is radially symmetric in x for all $t \geq 0$, we define $I_m^1(t)$ for $u(t)$ by (2.6). By (1.11) and (2.17), we have

$$\begin{aligned} \int_{|x| \geq m} |u(t, x)|^{p+1} dx &\leq \|u(t)\|_{L^\infty(|x| \geq m)}^{p-1} \|u(t)\|_2^2 \\ &\leq C m^{-(N-1)(p-1)/2} \|u(t)\|_{H^1}^{p+1} \leq C M_1^{p+1} m^{-(N-1)(p-1)/2} \end{aligned}$$

for all $t \geq 0$ and $m > 0$. Note that we assume $N \geq 2$. Thus, there exists $m_0 > 0$ such that

$$\sup_{t \geq 0} \left(\frac{N(p-1)}{p+1} \int_{|x| \geq m_0} |u(t, x)|^{p+1} dx + \frac{C_0}{m_0^2} \|u(t)\|_2^2 \right) < \delta.$$

Thus, by Lemmas 8 and 10, we have

$$\begin{aligned} &\frac{d}{dt} I_{m_0}^1(t) \\ &\geq -P(u(t)) - \left(\frac{N(p-1)}{p+1} \int_{|x| \geq m_0} |u(t, x)|^{p+1} dx + \frac{C_0}{m_0^2} \|u(t)\|_2^2 \right) \\ &\geq 2\delta - \delta = \delta \end{aligned}$$

for all $t \geq 0$. Therefore, we have $\lim_{t \rightarrow \infty} I_{m_0}^1(t) = \infty$. On the other hand, there exists a constant $C = C(m_0) > 0$ such that $I_{m_0}^1(t) \leq C \|\vec{u}(t)\|_X^2 \leq C M_1^2$ for all $t \geq 0$. This is a contradiction. Hence, for any $\lambda > 1$, the solution $u(t)$ of (1.1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ either blows up in finite time or exists for all $t \geq 0$ and $\limsup_{t \rightarrow \infty} \|\vec{u}(t)\|_X = \infty$. This completes the proof of Theorem 1 for the case $p \geq 1 + 4/N$. \square

Next, we consider the case where $p < 1 + 4/N$. For this case, we need a different variational characterization of the ground state ϕ_ω of (1.2) from that for the case $p \geq 1 + 4/N$. We define the functional

$$K_\omega^0(u) = \alpha(1 - \omega^2)\|u\|_2^2 + (\alpha + 2)\{\|\nabla u\|_2^2 - \frac{2}{p+1}\|u\|_{p+1}^{p+1}\},$$

and consider the constrained minimization problem

$$(2.18) \quad d_\omega^0 = \inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}, K_\omega^0(u) = 0\}$$

and the set

$$(2.19) \quad \mathcal{R}_\omega^0 = \{(u, v) \in X : (E - \omega Q)(u, v) < d_\omega^0, K_\omega^0(u) < 0\},$$

where $\alpha = 4/(p-1) - N > 0$. Note that

$$(2.20) \quad K_\omega^0(u) = 2\partial_\lambda J_\omega(\lambda^\beta u(\lambda \cdot))|_{\lambda=1}, \quad \beta = \frac{\alpha + N}{2} = \frac{2}{p-1}.$$

Lemma 11 *Let $N \geq 2$, $1 < p < 1 + 4/N$ and $\omega \in (-1, 1)$. Then, we have the following.*

- (i) $\frac{1 - \omega^2}{\alpha + 2}\|u\|_2^2 > d_\omega^0$ for all $u \in H^1(\mathbb{R}^N)$ satisfying $K_\omega^0(u) < 0$.
- (ii) The minimization problem (2.18) is attained at the ground state ϕ_ω of (1.2).
- (iii) $\lambda(\phi_\omega, i\omega\phi_\omega) \in \mathcal{R}_\omega^0$ for all $\lambda > 1$.

Proof. First, we note that

$$(2.21) \quad J_\omega(u) - \frac{1}{2(\alpha + 2)}K_\omega^0(u) = \frac{1 - \omega^2}{\alpha + 2}\|u\|_2^2,$$

$$(2.22) \quad d_\omega^0 = \inf\left\{\frac{1 - \omega^2}{\alpha + 2}\|u\|_2^2 : u \in H^1(\mathbb{R}^N) \setminus \{0\}, K_\omega^0(u) = 0\right\}.$$

(i) Let $u \in H^1(\mathbb{R}^N)$ satisfy $K_\omega^0(u) < 0$. Then, we have $u \neq 0$, and there exists $\lambda_1 \in (0, 1)$ such that $K_\omega^0(\lambda_1 u) = 0$. By (2.18), we have

$$d_\omega^0 \leq \frac{1 - \omega^2}{\alpha + 2}\|\lambda_1 u\|_2^2 < \frac{1 - \omega^2}{\alpha + 2}\|u\|_2^2.$$

(ii) Note that $d_\omega^0 \geq 0$ by (2.22). Let $\{u_j\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for (2.18). By considering the Schwarz symmetrization of u_j , we

can assume that $\{u_j\} \subset H_{rad}^1(\mathbb{R}^N)$. We refer to [2, Appendix A.III] for the definition and basic properties of the Schwarz symmetrization. By (2.22), we see that $\{u_j\}$ is bounded in $L^2(\mathbb{R}^N)$. Moreover, by $K_\omega^0(u_j) = 0$ and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} & (\alpha + 2)\|\nabla u_j\|_2^2 + \alpha(1 - \omega^2)\|u_j\|_2^2 \\ &= \frac{2(\alpha + 2)}{p + 1}\|u_j\|_{p+1}^{p+1} \leq C\|u_j\|_2^{p+1-\theta}\|\nabla u_j\|_2^\theta, \end{aligned}$$

where $\theta = (p - 1)N/2$. Since $p < 1 + 4/N$, we see that $\theta < 2$ and that $\{u_j\}$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, there exist a subsequence of $\{u_j\}$ (we still denote it by the same letter) and $w \in H_{rad}^1(\mathbb{R}^N)$ such that $u_j \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$ and $u_j \rightarrow w$ strongly in $L^{p+1}(\mathbb{R}^N)$. Here, we used the fact that the embedding $H_{rad}^1(\mathbb{R}^N) \hookrightarrow L_{rad}^q(\mathbb{R}^N)$ is compact for $2 < q < 2 + 4/(N - 2)$ (see [30]). Next, we show that $w \neq 0$. Suppose that $w = 0$. Then, by $K_\omega^0(u_j) = 0$ and the strong convergence $u_j \rightarrow 0$ in $L^{p+1}(\mathbb{R}^N)$, we see that $u_j \rightarrow 0$ in $H^1(\mathbb{R}^N)$. On the other hand, by $K_\omega^0(u_j) = 0$ and the Sobolev inequality, we have

$$\begin{aligned} & (\alpha + 2)\|\nabla u_j\|_2^2 + \alpha(1 - \omega^2)\|u_j\|_2^2 = \frac{2(\alpha + 2)}{p + 1}\|u_j\|_{p+1}^{p+1} \\ & \leq C\{(\alpha + 2)\|\nabla u_j\|_2^2 + \alpha(1 - \omega^2)\|u_j\|_2^2\}^{(p+1)/2}. \end{aligned}$$

Since $u_j \neq 0$, we have $\|u_j\|_{H^1} \geq C$ for some $C > 0$. This is a contradiction. Thus, we see that $w \in H^1(\mathbb{R}^N) \setminus \{0\}$. Therefore, by (2.21) and (2.22), we have

$$d_\omega^0 \leq \frac{1 - \omega^2}{\alpha + 2}\|w\|_2^2 \leq \liminf_{j \rightarrow \infty} \frac{1 - \omega^2}{\alpha + 2}\|u_j\|_2^2 = \liminf_{j \rightarrow \infty} J_\omega(u_j) = d_\omega^0,$$

and $K_\omega^0(w) \leq \liminf_{j \rightarrow \infty} K_\omega^0(u_j) = 0$. Moreover, by (i), we have $K_\omega^0(w) = 0$. Therefore, w attains (2.22) and (2.18). Since w attains (2.18), there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that

$$(2.23) \quad J'_\omega(w) = \frac{\eta}{2(\alpha + 2)}(K_\omega^0)'(w).$$

That is, w satisfies

$$(2.24) \quad -(1 - \eta)\Delta w + (1 - \omega^2)\left(1 - \frac{\alpha}{\alpha + 2}\eta\right)w - (1 - \eta)|w|^{p-1}w = 0$$

in $H^{-1}(\mathbb{R}^N)$. First, we show that $\eta < 1$. Suppose that $\eta \geq 1$. Then, by

(2.24) and $K_\omega^0(w) = 0$, we have

$$\begin{aligned}
0 &= (1 - \eta)\|\nabla w\|_2^2 + (1 - \omega^2)\left(1 - \frac{\alpha}{\alpha + 2}\eta\right)\|w\|_2^2 - (1 - \eta)\|w\|_{p+1}^{p+1} \\
&= \frac{(\eta - 1)(p - 1)}{2}\|\nabla w\|_2^2 + \frac{\alpha(p - 1)(1 - \omega^2)}{2(\alpha + 2)}\left\{\eta - 1 + \frac{4}{\alpha(p - 1)}\right\}\|w\|_2^2 \\
&\geq \frac{2(1 - \omega^2)}{\alpha + 2}\|w\|_2^2 > 0.
\end{aligned}$$

This is a contradiction. Thus, we have $\eta < 1$. Since we have

$$1 - \eta > 0, \quad (1 - \omega^2)\left(1 - \frac{\alpha}{\alpha + 2}\eta\right) > 0$$

in (2.24), by [6, Theorem 8.1.1], we have $x \cdot \nabla w \in H^1(\mathbb{R}^N)$. Therefore, by (2.23), we have

$$\begin{aligned}
0 &= K_\omega^0(w) = 2\partial_\lambda J_\omega(\lambda^\beta w(\lambda \cdot))|_{\lambda=1} = 2\langle J'_\omega(w), x \cdot \nabla w + \beta w \rangle \\
&= \frac{\eta}{\alpha + 2}\langle (K_\omega^0)'(w), x \cdot \nabla w + \beta w \rangle = \frac{\eta}{\alpha + 2}\partial_\lambda K_\omega^0(\lambda^\beta w(\lambda \cdot))|_{\lambda=1}
\end{aligned}$$

where $\beta = (\alpha + N)/2$. Moreover, by $K_\omega^0(w) = 0$, we have

$$\begin{aligned}
&\partial_\lambda K_\omega^0(\lambda^\beta w(\lambda \cdot))|_{\lambda=1} \\
&= \alpha^2(1 - \omega^2)\|w\|_2^2 + (\alpha + 2)^2\left\{\|\nabla w\|_2^2 - \frac{2}{p + 1}\|w\|_{p+1}^{p+1}\right\} \\
&= -2\alpha(1 - \omega^2)\|w\|_2^2 < 0.
\end{aligned}$$

Thus, we have $\eta = 0$. Therefore, w satisfies $J'(w) = 0$ and $K_\omega^2(w) = 0$, where

$$K_\omega^2(u) := \langle J'_\omega(u), u \rangle = \|\nabla u\|_2^2 + (1 - \omega^2)\|u\|_2^2 - \|u\|_{p+1}^{p+1}.$$

Since ϕ_ω attains

$$\inf\{J_\omega(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}, K_\omega^2(u) = 0\}$$

(see, e.g., [23][Lemma 3]), we have $J_\omega(\phi_\omega) \leq J_\omega(w)$. On the other hand, ϕ_ω satisfies $K_\omega^0(\phi_\omega) = 0$, we have $d_\omega^0 = J_\omega(w) \leq J_\omega(\phi_\omega)$. Hence, ϕ_ω attains (2.18).

(iii) The proof is similar to that of Lemma 9 (iii), and we omit it. \square

Lemma 12 *Suppose that $N \geq 2$, $1 < p < 1 + 4/N$ and $\omega \in (-1, 1)$. If $(u_0, u_1) \in \mathcal{R}_\omega^0$, then the solution $u(t)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ satisfies*

$$\frac{1 - \omega^2}{\alpha + 2}\|u(t)\|_2^2 > d_\omega^0, \quad t \in [0, T_{\max}).$$

Proof. The proof is similar to that for Lemma 10. We omit the details. \square

Proof of Theorem 1 for the case $p < 1 + 4/N$. Let $\lambda > 1$ be fixed and define

$$\begin{aligned}\delta_1 &= (\alpha + 2)\{d_\omega^0 - (E - \omega Q)(\lambda(\phi_\omega, i\omega\phi_\omega))\}, \\ \delta_2 &= \alpha\{\omega Q(\lambda(\phi_\omega, i\omega\phi_\omega)) - \frac{\omega^2(\alpha + 2)}{1 - \omega^2}d_\omega^0\},\end{aligned}$$

and $\delta = \delta_1 + \delta_2$. Then, by Lemma 11 (iii), we have $\delta_1 > 0$. Moreover, by Lemma 11 (ii) and (2.22), we have

$$\frac{\omega^2(\alpha + 2)}{1 - \omega^2}d_\omega^0 = \omega^2\|\phi_\omega\|_2^2 < \lambda^2\omega^2\|\phi_\omega\|_2^2 = \omega Q(\lambda(\phi_\omega, i\omega\phi_\omega)).$$

Thus, we have $\delta_2 > 0$ and $\delta > 0$. Suppose that the solution $u(t)$ of (1.1) with $\vec{u}(0) = \lambda(\phi_\omega, i\omega\phi_\omega)$ exists for all $t \in [0, \infty)$ and is uniformly bounded in X . Since $u(t)$ is radially symmetric in x for all $t \geq 0$, we define $I_m^2(t)$ for $u(t)$ by (2.7). As in the proof of Theorem 1 for the case $p \geq 1 + 4/N$, there exists $m_0 > 0$ such that

$$\sup_{t \geq 0} \left(\frac{N(p-1)}{p+1} \int_{|x| \geq m_0} |u(t, x)|^{p+1} dx + \frac{C_0}{m_0^2} \|u(t)\|_2^2 \right) < \delta.$$

Thus, by Lemma 8, we have

$$\frac{d}{dt} I_{m_0}^2(t) \geq -K(\vec{u}(t)) - \delta, \quad t \geq 0.$$

Here, recall that we assume $|\omega| \leq \omega_c$, so we have $1 - (\alpha + 1)\omega^2 \geq 0$. Thus, by (1.18) and Lemma 12, we have

$$\begin{aligned}& -K(\vec{u}(t)) \\ & \geq -2(\alpha + 2)(E - \omega Q)(\vec{u}(t)) + 2\alpha\omega Q(\vec{u}(t)) + 2\{1 - (\alpha + 1)\omega^2\}\|u(t)\|_2^2 \\ & \geq -2(\alpha + 2)(E - \omega Q)(\vec{u}(0)) + 2\alpha\omega Q(\vec{u}(0)) + 2\{1 - \omega^2 - \alpha\omega^2\}\frac{\alpha + 2}{1 - \omega^2}d_\omega^0 \\ & = 2\delta\end{aligned}$$

for all $t \geq 0$. Therefore, we have $(d/dt)I_{m_0}^2(t) \geq \delta$ for all $t \geq 0$, and $\lim_{t \rightarrow \infty} I_{m_0}^2(t) = \infty$. The rest of the proof is the same as in the proof of Theorem 1 for the case $p \geq 1 + 4/N$, and we omit the details. \square

Proof of Theorem 4. Let us first note that identity (1.18) contains the reason that in Theorem 4 we can allow any radially symmetric solutions of (1.2), unlike the case of Theorem 1 where we can treat only the ground state

of (1.2). Namely, when $\omega = \omega_c$ we have $1 - (\alpha + 1)\omega_c^2 = 0$, and therefore the identity (1.18) does not contain the norm $\|u\|_2^2$. Let us recall that in Theorem 1 we control this norm by using the variational characterization of the ground state.

Let $\varphi \in H^1(\mathbb{R}^N) \setminus \{0\}$ be a radially symmetric solution of (1.2) with $\omega = \omega_c$. Let $\lambda > 1$ and put

$$\delta = \alpha\omega_c Q(\lambda\varphi, i\omega_c\varphi) - (\alpha + 2)(E - \omega_c Q)(\lambda\varphi, i\omega_c\varphi).$$

Since $J'_{\omega_c}(\varphi) = 0$, we have $(E - \omega_c Q)(\lambda\varphi, i\omega_c\varphi) = J_{\omega_c}(\lambda\varphi) < J_{\omega_c}(\varphi)$ for $\lambda > 1$. Moreover, we have $\omega_c Q(\lambda\varphi, i\omega_c\varphi) = \omega_c^2 \lambda^2 \|\varphi\|_2^2 > \omega_c^2 \|\varphi\|_2^2$ for $\lambda > 1$. Thus, we have

$$\delta > \alpha\omega_c^2 \|\varphi\|_2^2 - (\alpha + 2)J_{\omega_c}(\varphi) = -\frac{1}{2}K_{\omega_c}^0(\varphi) - \{1 - (\alpha + 1)\omega_c^2\} \|\varphi\|_2^2.$$

By [6, Theorem 8.1.1], we have $x \cdot \nabla\varphi \in H^1(\mathbb{R}^N)$. Therefore, by (2.20) and by $J'_{\omega_c}(\varphi) = 0$, we have

$$K_{\omega_c}^0(\varphi) = 2\langle J'_{\omega_c}(\varphi), x \cdot \nabla\varphi + \beta\varphi \rangle = 0.$$

Moreover, since $(\alpha + 1)\omega_c^2 = 1$, we have $\delta > 0$. Suppose that the solution $u(t)$ of (1.1) with $\vec{u}(0) = \lambda(\varphi, i\omega_c\varphi)$ exists for all $t \in [0, \infty)$ and is uniformly bounded in X . Since $u(t)$ is radially symmetric in x for all $t \geq 0$, we define $I_m^2(t)$ for $u(t)$ by (2.7). As in the proof of Theorem 1 for the case $p \geq 1 + 4/N$, there exists $m_0 > 0$ such that

$$\sup_{t \geq 0} \left(\frac{N(p-1)}{p+1} \int_{|x| \geq m_0} |u(t, x)|^{p+1} dx + \frac{C_0}{m_0^2} \|u(t)\|_2^2 \right) < \delta.$$

Thus, by Lemma 8, we have

$$\frac{d}{dt} I_{m_0}^2(t) \geq -K(\vec{u}(t)) - \delta, \quad t \geq 0.$$

Moreover, by (1.18) and $(\alpha + 1)\omega_c^2 = 1$, we have

$$\begin{aligned} & -K(\vec{u}(t)) \\ & \geq -2(\alpha + 2)(E - \omega_c Q)(\vec{u}(t)) + 2\alpha\omega_c Q(\vec{u}(t)) + 2\{1 - (\alpha + 1)\omega_c^2\} \|u(t)\|_2^2 \\ & \geq -2(\alpha + 2)(E - \omega_c Q)(\vec{u}(0)) + 2\alpha\omega_c Q(\vec{u}(0)) = 2\delta \end{aligned}$$

for all $t \geq 0$. Therefore, we have $(d/dt)I_{m_0}^2(t) \geq \delta$ for all $t \geq 0$, and $\lim_{t \rightarrow \infty} I_{m_0}^2(t) = \infty$. On the other hand, there exists a constant $C = C(m_0) > 0$ such that $I_{m_0}^2(t) \leq C\|\vec{u}(t)\|_X^2 \leq C$ for all $t \geq 0$. This is

a contradiction. Therefore, for any $\lambda > 1$, the solution $u(t)$ of (1.1) with $\vec{u}(0) = \lambda(\varphi, i\omega_c\varphi)$ either blows up in finite time or exists for all $t \geq 0$ and $\limsup_{t \rightarrow \infty} \|\vec{u}(t)\|_X = \infty$. Finally, by Lemma 2, if $u(t)$ exists for all $t \geq 0$, then $\sup_{t \geq 0} \|u(t)\|_X < \infty$. Hence, $u(t)$ blows up in finite time. This completes the proof. \square

Proof of Lemma 2. By Proposition 3.1 and Lemma 3.5 in [5], we have

$$(2.25) \quad \sup_{t \geq 0} \|u(t)\|_2 < \infty,$$

$$(2.26) \quad \sup_{t \geq 0} \int_t^{t+1} \|\vec{u}(s)\|_X^2 ds < \infty.$$

By (2.26) and the conservation of energy E , we have

$$(2.27) \quad C_1 := \sup_{t \geq 0} \int_t^{t+1} \|u(s)\|_{p+1}^{p+1} ds < \infty.$$

Note that the estimates (2.25), (2.26) and (2.27) hold true for $1 < p < 1 + 4/(N - 2)$. In what follows, we use an argument in Merle and Zaag [18]. First, for $r = (p + 3)/2$, we show

$$(2.28) \quad \sup_{t \geq 0} \|u(t)\|_r < \infty.$$

Indeed, by (2.27) and the mean value theorem, for any $t \geq 0$ there exists $\tau(t) \in [t, t + 1]$ such that

$$(2.29) \quad \|u(\tau(t))\|_{p+1}^{p+1} = \int_t^{t+1} \|u(s)\|_{p+1}^{p+1} ds \leq C_1.$$

Since $2 < r < p + 1$, it follows from (2.25) and (2.29) that $\sup_{t \geq 0} \|u(\tau(t))\|_r < \infty$. Moreover, for any $t \geq 0$, we have

$$\begin{aligned} \|u(t)\|_r^r - \|u(\tau(t))\|_r^r &= \int_{\tau(t)}^t \frac{d}{ds} \|u(s)\|_r^r ds \\ &\leq C \int_t^{t+1} \int_{\mathbb{R}^N} |u(s, x)|^{r-1} |\partial_s u(s, x)| dx ds \\ &\leq C \int_t^{t+1} (\|u(s)\|_{2(r-1)}^{2(r-1)} + \|\partial_s u(s)\|_2^2) ds. \end{aligned}$$

Since $2(r - 1) = p + 1$, by (2.26), (2.27) and $\sup_{t \geq 0} \|u(\tau(t))\|_r < \infty$, we have (2.28). Next, by the Gagliardo-Nirenberg inequality, we have

$$\|u(t)\|_{p+1} \leq C \|u(t)\|_r^{1-\theta} \|\nabla u(t)\|_2^\theta,$$

where

$$\frac{1}{p+1} = \theta\left(\frac{1}{2} - \frac{1}{N}\right) + \frac{1-\theta}{r}.$$

Since we assume $p < 1 + 4/(N-1)$, we have $(p+1)\theta < 2$. Thus, by (2.28), there exists a constant $C_2 > 0$ such that

$$\frac{2}{p+1} \|u(t)\|_{p+1}^{p+1} \leq C_2 + \frac{1}{2} \|\nabla u(t)\|_2^2, \quad t \geq 0.$$

Moreover, by the conservation of energy E , for any $t \geq 0$ we have

$$\begin{aligned} \|\vec{u}(t)\|_X^2 &= 2E(\vec{u}(0)) + \frac{2}{p+1} \|u(t)\|_{p+1}^{p+1} \\ &\leq 2E(\vec{u}(0)) + C_2 + \frac{1}{2} \|\nabla u(t)\|_2^2, \end{aligned}$$

which implies $\|\vec{u}(t)\|_X^2 \leq 4E(\vec{u}(0)) + 2C_2$. This completes the proof. \square

We conclude this section with the proof of Theorem A.

Proof of Theorem A (due to Kenji Nakanishi). Following the proof of Theorem 1, take the radially symmetric solution $u(t, r)$ ($r = |x|$) starting from $(u(0), \partial_t u(0)) = \lambda(\phi_\omega, i\omega\phi_\omega)$ with $\lambda > 1$, and assume by contradiction that it exists for all $t \geq 0$. Then Cazenave's estimate (2.26) implies that there exists $M < \infty$ such that for all $T > 0$

$$(2.30) \quad \int_T^{T+1} \int_{\mathbb{R}^N} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 dx dt \leq M.$$

Hence for any positive integer j , there exists $T_j \in [j-1, j]$ such that

$$\int_{\mathbb{R}^N} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 dx|_{t=T_j} \leq M.$$

By Lemmas 8, 9 and 10, there exists $\delta > 0$ such that for any $m > 1$ and $t > 0$ we have

$$\frac{d}{dt} I_m^1(t) \geq 2\delta - R_m(t), \quad R_m(t) := \frac{N(p-1)}{p+1} \int_{|x| \geq m} |u|^{p+1} dx + \frac{C}{m^2} \|u(t)\|_2^2,$$

where I_m^1 is defined by (2.6). Here and below C is a positive constant, which may depend only on p and N . Integrating in t , we get

$$I_m^1(T_{j+2}) - I_m^1(T_j) \geq 2\delta - \int_{T_j}^{T_{j+2}} R_m(t) dt,$$

since $T_{j+2} - T_j \geq 1$. Notice that (2.30) is enough to control the error term R_m uniformly in j . To see this, let $\chi(t, r) \in C^\infty(\mathbb{R}^2)$ satisfy $\chi(t, r) = 1$ when $|t| \leq 2$ and $|r| \geq 1$, and $\chi(t, r) = 0$ if $|t| \geq 4$ or $|r| \leq 1/2$. For any $m > 1$ and $T > 4$, let $v(t, r) = \chi(t - T, r/m)u(t, |r|)$. Then we have

$$\begin{aligned} & \int_{\mathbb{R}^2} |\partial_t v|^2 + |\partial_r v|^2 + |v|^2 dr dt \\ & \leq Cm^{1-N} \int_{T-4}^{T+4} \int_{\mathbb{R}^N} |\partial_t u|^2 + |\nabla u|^2 + |u|^2 dx dt \leq 8Cm^{1-N}M. \end{aligned}$$

Hence the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^{p+1}(\mathbb{R}^2)$ implies that

$$\begin{aligned} \int_{T-2}^{T+2} \int_{|x| \geq m} |u|^{p+1} dx dt & \leq C \sum_{j=0}^{\infty} \int_{T-2}^{T+2} (2^j m)^{N-1} \int_{r \geq 2^j m} |u|^{p+1} dr dt \\ & \leq Cm^{-(p-1)(N-1)/2} M^{(p+1)/2}. \end{aligned}$$

Therefore choosing m sufficiently large, we obtain

$$I_m^1(T_{j+2}) - I_m^1(T_j) \geq \delta$$

for all $j \geq 4$, which contradicts the global bound

$$I_m^1(T_j) \leq Cm \int_{\mathbb{R}^N} |\partial_t u|^2 + |\partial_r u|^2 + |u|^2 dx|_{t=T_j} \leq CmM.$$

□

3 Proof of Theorems for KGZ system

In this section, we prove Theorems 5 and 6.

Proof of Theorem 5. Let $\lambda > 1$ and put

$$\begin{aligned} \tilde{d}_\omega &= (H - \omega Q)(\phi_\omega, i\omega\phi_\omega, -|\phi_\omega|^2, 0), \\ \delta &= N\{\tilde{d}_\omega - (H - \omega Q)(\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)\}, \end{aligned}$$

where H and Q are defined by (1.19) and (1.6), respectively. In the same way as in Lemma 9 (iii), we see that $\delta > 0$. Suppose that the solution $\mathbf{u}(t)$ of (1.3)-(1.4) with $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ exists globally and satisfies $M := \sup_{t \geq 0} \|\mathbf{u}(t)\|_Y < \infty$. Note that since the initial data is radially symmetric, the solution $\mathbf{u}(t)$ is also radially symmetric for all $t \geq 0$. Following

Merle [17], we introduce the function $w(t) := -(-\Delta)^{-1}\partial_t n(t)$, and for $m > 0$ we consider the function

$$\tilde{I}_m(t) = I_m^1(t) + \frac{1}{c_0^2} \int_{\mathbb{R}^N} \Psi_m n(t) \partial_r w(t) dx,$$

where $I_m^1(t)$ is defined by (2.6) and Φ_m and Ψ_m are given by (2.1). Note that since $\partial_t n(t) \in \dot{H}^{-1}(\mathbb{R}^N)$, we see that $w(t) \in \dot{H}^1(\mathbb{R}^N)$ and $\|\partial_t n\|_{\dot{H}^{-1}} = \|\nabla w\|_2$. By the same computations as in Lemma 8, we have

$$\begin{aligned} -\frac{d}{dt} \tilde{I}_m(t) &= 2 \int_{\mathbb{R}^N} \Psi'_m |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \Phi_m (n^2 + 2|u|^2 n) dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \Delta \Phi_m |u|^2 dx + \frac{1}{2c_0^2} \int_{\mathbb{R}^N} \left(\Psi'_m - \frac{N-1}{r} \Psi_m \right) |\nabla w|^2 dx. \end{aligned}$$

By Lemma 7, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \Psi'_m |\nabla u|^2 dx &\leq \|\nabla u(t)\|_2^2, \\ -\frac{1}{2} \int_{\mathbb{R}^N} \Delta \Phi_m |u|^2 dx &\leq \frac{C_1}{m^2} \|u(t)\|_2^2 \leq \frac{C_1 M^2}{m^2}, \\ \int_{\mathbb{R}^N} \left(\Psi'_m - \frac{N-1}{r} \Psi_m \right) |\nabla w|^2 dx &\leq \|\nabla w(t)\|_2^2 = \|\partial_t n(t)\|_{\dot{H}^{-1}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \Phi_m (n^2 + 2|u|^2 n) dx \\ &= \int_{\mathbb{R}^N} \Phi_m (n + |u|^2)^2 dx - \int_{\mathbb{R}^n} N |u|^4 dx + \int_{\mathbb{R}^n} (N - \Phi_m) |u|^4 dx \\ &\leq N \|n + |u|^2\|_2^2 - N \|u\|_4^4 + \int_{|x| \geq m} (N - \Phi_m) |u|^4 dx, \end{aligned}$$

and by (1.11) we have

$$\begin{aligned} \frac{1}{2} \int_{|x| \geq m} (N - \Phi_m) |u|^4 dx &\leq C \|u(t)\|_{L^\infty(|x| \geq m)}^2 \|u(t)\|_2^2 \\ &\leq \frac{C_2}{m^{N-1}} \|u(t)\|_{H^1}^4 \leq \frac{C_2 M^4}{m^{N-1}}. \end{aligned}$$

Therefore, we have

$$(3.1) \quad -\frac{d}{dt} \tilde{I}_m(t) \leq \tilde{P}(\mathbf{u}(t)) + \frac{C_1 M^2}{m^2} + \frac{C_2 M^4}{m^{N-1}}$$

for all $t \geq 0$, where we put

$$\tilde{P}(u, v, n, \nu) = 2\|\nabla u\|_2^2 - \frac{N}{2}\|u\|_4^4 + \frac{N}{2}\|n + |u|^2\|_2^2 + \frac{1}{2c_0^2}\|\nu\|_{\dot{H}^{-1}}^2.$$

Note that

$$\begin{aligned} & (H - \omega Q)(u, v, n, \nu) - \frac{1}{2N}\tilde{P}(u, v, n, \nu) \\ &= \frac{1}{2}\|v - i\omega u\|_2^2 + \left(1 - \frac{1}{N}\right)\frac{1}{4c_0^2}\|\nu\|_{\dot{H}^{-1}}^2 + \left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1 - \omega^2}{2}\|u\|_2^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1 - \omega^2}{2}\|u\|_2^2. \end{aligned}$$

Using this inequality, in the same way as in Lemmas 9 and 10, we see that

$$(3.2) \quad -\tilde{P}(\mathbf{u}(t)) \geq 2N\{\tilde{d}_\omega - (H - \omega Q)(\mathbf{u}(0))\} = 2\delta$$

holds for all $t \geq 0$. Therefore, taking $m_1 > 0$ such that

$$\frac{C_1 M^2}{m_1^2} + \frac{C_2 M^4}{m_1^{N-1}} < \delta,$$

by (3.1) and (3.2), we have $(d/dt)\tilde{I}_{m_1}(t) \geq \delta$ for all $t \geq 0$, and $\lim_{t \rightarrow \infty} \tilde{I}_{m_1}(t) = \infty$. The rest of the proof is the same as in the proof of Theorem 1 for the case $p \geq 1 + 4/N$, and we omit the details. \square

Proof of Theorem 6. Let $\lambda > 1$. Suppose that the solution $\mathbf{u}(t)$ of (1.3)-(1.4) with $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ exists globally. By the assumption $|\omega| < 1/\sqrt{3}$, we can take α such that $2\omega^2/(1 - \omega^2) < \alpha < 1$. For such an α , we consider a function defined by

$$I_\alpha(t) = \frac{1}{2} \left\{ \|u(t)\|_2^2 + \frac{\alpha}{c_0^2} \|n(t)\|_{\dot{H}^{-1}}^2 \right\}.$$

Note that since $n(0) = -\lambda^2|\phi_\omega|^2 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \subset \dot{H}^{-1}(\mathbb{R}^3)$ and $\partial_t n \in C([0, \infty); \dot{H}^{-1}(\mathbb{R}^3))$, we see that $n \in C^1([0, \infty); \dot{H}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$. Then, we have

$$\begin{aligned} \frac{d}{dt} I_\alpha(t) &= \operatorname{Re} \langle u(t), \partial_t u(t) \rangle_{L^2} + \frac{\alpha}{c_0^2} \langle n(t), \partial_t n(t) \rangle_{\dot{H}^{-1}} \\ &= \operatorname{Re} \langle u(t), \partial_t u(t) - i\omega u(t) \rangle_{L^2} + \frac{\alpha}{c_0^2} \langle n(t), \partial_t n(t) \rangle_{\dot{H}^{-1}}, \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} I_\alpha(t) &= \|\partial_t u(t)\|_2^2 + \frac{\alpha}{c_0^2} \|\partial_t n(t)\|_{\dot{H}^{-1}}^2 - \|\nabla u(t)\|_2^2 - \|u(t)\|_2^2 \\ &\quad - \alpha \|n(t)\|_2^2 - (1 + \alpha) \int_{\mathbb{R}^3} |u(t, x)|^2 n(t, x) dx. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\frac{d^2}{dt^2} I_\alpha(t) + 2(1 + \alpha)(H - \omega Q)(\mathbf{u}(0)) - 2\omega Q(\mathbf{u}(0)) \\ &= (2 + \alpha) \|\partial_t u(t) - i\omega u(t)\|_2^2 + \left(2 + \frac{1 - \alpha}{2\alpha}\right) \frac{\alpha}{c_0^2} \|\partial_t n(t)\|_{\dot{H}^{-1}}^2 \\ &\quad + K_{\omega, \alpha}(u(t), n(t)), \end{aligned}$$

where we put

$$K_{\omega, \alpha}(u, n) = \alpha \left\{ \|\nabla u\|_2^2 + \left(1 - \omega^2 - \frac{2}{\alpha}\omega^2\right) \|u\|_2^2 + \frac{1 - \alpha}{2\alpha} \|n\|_2^2 \right\}.$$

Here, we define

$$\begin{aligned} J_\omega(u, n) &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1 - \omega^2}{2} \|u\|_2^2 + \frac{1}{4} \|n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} |u(x)|^2 n(x) dx, \\ K_{\omega, \alpha}^1(u, n) &= \partial_\lambda J_\omega(\lambda u, \lambda^{2\alpha} n)|_{\lambda=1} \\ &= \|\nabla u\|_2^2 + (1 - \omega^2) \|u\|_2^2 + \alpha \|n\|_2^2 + (1 + \alpha) \int_{\mathbb{R}^3} |u|^2 n dx, \\ K_{\omega, \alpha}^2(u, n) &= 2\partial_\lambda J_\omega(\lambda^{(1-\alpha)/\alpha} u(\cdot/\lambda), n(\cdot/\lambda))|_{\lambda=1} \\ &= \frac{2 - \alpha}{\alpha} \|\nabla u\|_2^2 + \frac{2 + \alpha}{\alpha} (1 - \omega^2) \|u\|_2^2 \\ &\quad + \frac{3}{2} \|n\|_2^2 + \frac{2 + \alpha}{\alpha} \int_{\mathbb{R}^3} |u|^2 n dx, \end{aligned}$$

and put

$$\begin{aligned} J_{\omega, \alpha}^1(u, n) &= J_\omega(u, n) - \frac{1}{2(1 + \alpha)} K_{\omega, \alpha}^1(u, n) \\ &= \frac{\alpha}{1 + \alpha} \left\{ \frac{1}{2} \|\nabla u\|_2^2 + \frac{1 - \omega^2}{2} \|u\|_2^2 + \frac{1 - \alpha}{4\alpha} \|n\|_2^2 \right\}, \\ J_{\omega, \alpha}^2(u, n) &= J_\omega(u, n) - \frac{\alpha}{2(2 + \alpha)} K_{\omega, \alpha}^2(u, n) \\ &= \frac{\alpha}{2 + \alpha} \left\{ \|\nabla u\|_2^2 + \frac{1 - \alpha}{2\alpha} \|n\|_2^2 \right\}, \\ \theta &= 1 - \frac{2\omega^2}{(1 - \omega^2)\alpha}. \end{aligned}$$

Then, we have $0 < \theta < 1$ and

$$K_{\omega,\alpha}(u, n) = 2(1 + \alpha)\theta J_{\omega,\alpha}^1(u, n) + (2 + \alpha)(1 - \theta)J_{\omega,\alpha}^2(u, n).$$

Moreover, in a similar way as in Lemmas 3 and 4 in [23], we can prove that $J_{\omega,\alpha}^j(u(t), n(t)) \geq \tilde{d}_\omega$ for all $t \geq 0$ and $j = 1, 2$. Therefore, we have

$$\begin{aligned} K_{\omega,\alpha}(u(t), n(t)) &\geq \{2(1 + \alpha)\theta + (2 + \alpha)(1 - \theta)\} \tilde{d}_\omega \\ &= 2 \left(1 + \alpha - \frac{\omega^2}{1 - \omega^2} \right) \tilde{d}_\omega \end{aligned}$$

for all $t \geq 0$. Moreover, since we have $\tilde{d}_\omega = (1 - \omega^2)\|\phi_\omega\|_2^2$, putting $\beta = \min\{2 + \alpha, 2 + (1 - \alpha)/2\alpha\}$, we have

$$\begin{aligned} \frac{d^2}{dt^2} I_\alpha(t) &\geq \beta \left\{ \|\partial_t u(t) - i\omega u(t)\|_2^2 + \frac{\alpha}{c_0^2} \|\partial_t n(t)\|_{\dot{H}^{-1}}^2 \right\} \\ &\quad + 2(1 + \alpha) \{ \tilde{d}_\omega - (H - \omega Q)(\mathbf{u}(0)) \} + 2\omega Q(\mathbf{u}(0)) - 2\omega^2 \|\phi_\omega\|_2^2 \end{aligned}$$

for all $t \geq 0$. Since $\beta > 2$, $(H - \omega Q)(\mathbf{u}(0)) < \tilde{d}_\omega$ and $\omega Q(\mathbf{u}(0)) > \omega^2 \|\phi_\omega\|_2^2$ for all $\lambda > 1$, by the standard concavity argument, we see that there exists $T_1 \in (0, \infty)$ such that $\lim_{t \rightarrow T_1-0} I_\alpha(t) = \infty$. This is a contradiction. Hence, for all $\lambda > 1$, the solution $\mathbf{u}(t)$ of (1.3)-(1.4) with $\mathbf{u}(0) = (\lambda\phi_\omega, \lambda i\omega\phi_\omega, -\lambda^2|\phi_\omega|^2, 0)$ blows up in finite time. This completes the proof. \square

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