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Critical Exponent for a Nonlinear Wave Equation with Damping

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1. INTRODUCTION

This article is concerned with the Cauchy problem for the dissipative nonlinear wave equation

$$\Box u + u_t = |u|^p, \qquad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n, \tag{1.1}$$

$$u|_{t=0} = \varepsilon u_0, \qquad u_t|_{t=0} = \varepsilon u_1, \tag{1.2}$$

where $\Box = \partial_t^2 - \Delta_x$ is the wave operator, $\varepsilon > 0$, and (u_0, u_1) are compactly supported data from the energy space:

$$u_0 \in H^1(\mathbf{R}^n), \qquad u_1 \in L^2(\mathbf{R}^n),$$
$$\operatorname{supp} u_i \subset B(K) \equiv \{x \in \mathbf{R}^n : |x| < K\}, \qquad i = 0, 1.$$

We study questions of global existence, blow-up and asymptotic behavior as $t \to \infty$ for solutions of (1.1), (1.2). Our interest is focused on the so-called critical exponent $p_c(n)$, which is the number defined by the following property:

> If $p_c(n) < p$, then all small data solutions of (1.1), (1.2) are global, while if $1 < p < p_c(n)$ all solutions of (1.1), (1.2) with data positive on average blow up infinite time regardless of the smallness of the data.

It is well known that if the damping is missing, the critical exponent for the nonlinear wave equation $\Box u = |u|^p$ is the positive root $p_0(n)$ of the equation $(n-1)\,p^2 - (n+1)\,p - 2 = 0$, where $n \geqslant 2$ is the space dimension (for $p_0(1) = \infty$, see Sideris [14]). The proof of this fact, known as Strauss' conjecture [17], took more than 20 years of effort, beginning with Glassey

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[6], John [9], Sideris [17], Choquet-Bruhat [2], [3], Zhou [21], Agemi, Kubota and Takmura [1], and ending recently with Lindblad and Sogge [12], Georgiev, Lindblad, and Sogge [5] and Tataru [20]. Global existence for the damped wave equation (1.1), for $p>1+4/n$ was proved by Nakao and Ono in [14].

In this paper we solve the critical exponent problem for the wave Eq. (1.1), (1.2) which involves not only a source but also a linear damping term. We show that the influence of the damping is powerful enough to shift the critical exponent $p_0(n)$ of the wave equation to the left, i.e. the critical exponent $p_c(n)$ for Eq. (1.1) is strictly less than $p_0(n)$.

The main result of this work is that the critical exponent $p_c(n)$ of the damped wave Eq. (1.1), (1.2) is exactly

$$p_c(n)=1+2/n;$$

thus, $p_c(n)<p_0(n)$.

Our global existence result can be stated as follows.

THEOREM 1.1. *Let $p_c(n)<p\leqslant n/(n-2)$ for $n\geqslant 3$, and $p_c(n)<p<\infty$ for $n=1, 2$. There exists $\varepsilon_0>0$ such that problem* (1.1), (1.2) *admits a unique global solution*

$$u\in C(\mathbf{R}_+, H^1(\mathbf{R}^n)), \qquad u_t\in C(\mathbf{R}_+, L^2(\mathbf{R}^n)),$$

for each $\varepsilon<\varepsilon_0$.

The blow-up result is complementary to Theorem 1.1 except for the critical case $p=p_c(n)$.

THEOREM 1.2. *Let $1<p<1+2/n$. If*

$$c_i\equiv\int u_i(x)\,dx>0, \qquad i=0, 1,$$

then the solution of (1.1), (1.2) *does not exist globally, for any $\varepsilon>0$.*

It should be pointed out that the critical exponent $p_c(n)=1+2/n$ of the damped wave Eq. (1.1), (1.2) is exactly equal to Fujita's critical exponent $p_1(n)$ for the nonlinear heat equation $u_t-\Delta u=|u|^p$. This indicates that the damping term drastically changes the asymptotic behavior of the solutions of the wave equation. In other words, these seem to behave more like solutions of the heat equation at large times.

In this study we have also discovered an interesting phenomenon caused by the presence of the damping term: *The support of the solution of* (1.1), (1.2) *with $p>p_c(n)$ is strongly suppressed by the damping*, *so that the solution is concentrated in a ball much smaller than* $|x|\leqslant t+K$. (The constant

K is the radius of the support of the initial data: $|x| < K$.) More precisely, we establish the following estimates.

THEOREM 1.3. *Let $p > 1 + 2/n$. Then the asymptotic behavior of any small data global solution of the Cauchy problem* (1.1), (1.2) *is given by*

$$\|Du(t,\cdot)\|_{L^2(R^n \setminus B(t^{1/2+\delta}))} = \mathcal{O}(e^{-t^{2\delta}/4}), \qquad t \to \infty, \tag{1.3}$$

that is the solution decays exponentially outside every ball $B(t^{1/2+\delta})$, $\delta > 0$. Moreover, the total energy decays at the rate of the linear equation, namely

$$\|Du(t,\cdot)\|_{L^2(R^n)} = \mathcal{O}(t^{-n/4-1/2}), \qquad t \to \infty, \tag{1.4}$$

where $D = (\frac{\partial}{\partial t}, \nabla_x)$.

It is important to note that none of the classical techniques for the critical exponent for the wave equation works for the damped wave Eq. (1.1), (1.2), so one is forced to find other techniques.

In Section 2 we prove the global existence Theorem 1.1 for small data solutions. We use a weighted energy with the weight function $e^{\psi(t,x)}$, where $\psi(t,x)$ behaves , roughly speaking, like $|x|^2/4t$. The explanation of why this "strange" weight function occurs is very natural. Namely, it is related to the form of the fundamental solution $S_2(t,x)$ of $\Box + \partial_t$ (cf. [10], Chapter 10), that is

$$S_2(t,x) = \begin{cases} C_n e^{-t/2}(\Box - 1/4)^{n/2-1} H(t)\, H(t^2 - |x|^2)\, I_0(\tfrac{1}{2}\sqrt{t^2 - |x|^2}), \\ \qquad n \text{ even}, \\ C_n e^{-t/2}(\Box - 1/4)^{n/2-1/2} H(t)\, H(t^2 - |x|^2)\, \sinh(\tfrac{1}{2}\sqrt{t^2 - |x|^2}), \\ \qquad n \text{ odd}. \end{cases}$$

Here C_n is a constant, while I_0 is the modified Bessel function of order 0 and H is the Heaviside function

$$H(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases} \tag{1.5}$$

Note that $I_0(\rho/2) = e^{\rho/2}/\sqrt{\pi\rho}\,(1 + \mathcal{O}(\frac{1}{\rho}))$ as $\rho = \sqrt{t^2 - |x|^2} \to \infty$ (cf. [13]). Thus, $S_2(t,x)$ has a factor $e^{-t/2 + \sqrt{t^2 - |x|^2}/2}$ which, as $t \to \infty$, is asymptotic to $e^{-|x|^2/4t}$. The same weight is suggested by the fundamental solution of the linear heat equation, in view of the so called diffusion phenomenon. It is known that the solutions of the heat equation and the solutions of the linear damped wave equation have similar asymptotic behavior, cf. [7], [8].

We gain the following benefits by using this weighted energy:

1. The damping strongly suppresses the support of the solution of the linear equation $\Box u + u_t = 0$. Namely, the solution decays exponentially outside the set $|x| \leqslant t^{1/2+\delta}$, for any $\delta > 0$.

2. By using weighted estimates we are able to gain decay and prove the global existence Theorem 1.1. Roughly speaking, the Sobolev constant in the usual Sobolev estimates is like t, while the corresponding Sobolev constant for the weighted estimate is like $t^{1/2+\delta}$.

3. We find the damping suppresses the support of the solution of the nonlinear Eq. (1.1), (1.2) when $p > 1 + 2/n$, in the sense that the solution decays exponentially in the outside of the ball $|x| \leqslant t^{1/2+\delta}$, for any $\delta > 0$.

In Section 3 we prove the blow-up Theorem 1.2. At the beginning, we show how a part of the blow-up range, namely $1 < p < 1 + 1/n$, can be treated in a relatively easy way by considering an appropriate average, and showing that this average blows up as the solution of some nonlinear ordinary differential inequality. The rest of the blow-up region, namely $1 + 1/n \leqslant p < 1 + 2/n$, requires a greater deal of effort. First, we discuss the nature of the difficulties and the possible ways to overcome them. The idea of Sideris to solve the positivity problem for the wave operator $\Box$, when $n > 3$, by introducing a space-time average, is difficult to implement in our situation, on account of the very complicated explicit expression for the fundamental solution. (See formula for $S_2(t, x)$ above.) For low space dimension $n \leqslant 3$, the situation does not seem more promising: the fundamental solution of the operator $\Box + \partial_t$ is now positive, but still not simple to deal with. To establish the counterparts of John's and Glassey's pointwise estimates for the wave equation would be possible but highly non-trivial. Thus, in any space dimension, to derive sharp lower estimates for the solution of (1.1), (1.2) seems to be a complicated task.

The main idea for this range is to consider as an average the convolution $S_{2k} * Hu$, where u is the solution of (1.1), (1.2), while $S_{2k} \in \mathscr{D}'(\mathbf{R}_+ \times \mathbf{R}^n)$ is the fundamental solution of the operator $(\Box + \partial_t)^k$. Considering high powers $k > (n+1)/2$, we gain both regularity and positivity of S_{2k}, which are crucial to carry out the lower estimates in obtaining blow-up. The usage of this convolution is paid by precise asymptotic analysis of the space-time average $S_{2k} * Hu$.

Note. After the paper was completed the authors learned that the critical exponent problem for (1.1) when $n = 1, 2$ was solved in the paper of Li and Zhou [11].

2. GLOBAL EXISTENCE

The local well-posedness of the problem studied is well known. The following classical result can be found in Strauss [19].

PROPOSITION 2.1 *Assume that* $p \in [1, \frac{n}{n-2}]$, *if* $n \geqslant 3$, *and* $p \in [1, \infty)$, *if* $n = 1, 2$. *Then the problem* (1.1), (1.2) *possesses a unique local solution* u *such that*

$$u \in C([0, T), H^1(\mathbf{R}^n)), \qquad u_t \in C([0, T), L^2(\mathbf{R}^n)),$$

$$\operatorname{supp} u(t, \cdot) \subset B(t+K).$$

Here $T > 0$ *depends on the norm* $\|Du(0)\|_{L^2}$. *Moreover, the solution can be continued beyond the interval* $[0, T)$ *if* $\sup_{[0, T)} \|Du(t)\|_{L^2} < \infty$.

In view of this result, global existence of a solution follows from the boundedness of its energy at all times. To derive such a priori bounds, our basic tool is a new weighted energy estimate. We begin with the identity

$$\begin{aligned} e^{2\psi} u_t(\Box u + u_t) = & \frac{d}{dt}\left(\frac{e^{2\psi}}{2}(|u_t|^2 + |\nabla u|^2)\right) \\ & - \operatorname{div}(e^{2\psi} u_t \nabla u) \\ & - \frac{e^{2\psi}}{\psi_t}(\psi_t \nabla u - u_t \nabla \psi)^2, \end{aligned} \tag{2.1}$$

which holds for each function $\psi(t, x)$ that solves the first order equation

$$\psi_t = \psi_t^2 - |\nabla \psi|^2. \tag{2.2}$$

In order to get good estimates, we would additionally like to have $\psi_t < 0$. It turns out that functions satisfying both conditions exist and have a simple form: for instance, we can choose

$$\psi(t, x) = \tfrac{1}{2}(t + K - \sqrt{(t+K)^2 - |x|^2}), \qquad |x| < t + K, \tag{2.3}$$

which not only satisfies the above conditions, but has the advantage of being regular on the support of the solution. We also note that, since

$$\sqrt{(t+K)^2 - |x|^2} \leqslant t + K - |x|^2/[2(t+K)],$$

the function ψ satisfies the inequality

$$\psi(t, x) \geqslant \frac{|x|^2}{4(t+K)}. \tag{2.4}$$

This leads to an interesting phenomenon, best observable in the case of the linear equation $\Box u + u_t = 0$. Assuming $u \in C^2([0, T), L^2(\mathbf{R}^n))$, in addition, and integrating (2.1) over the strip $[0, t] \times \mathbf{R}^n$, (recall that $\psi_t < 0$) we have

$$\int e^{2\psi(t, x)} \, |Du(t, x)|^2 \, dx \leqslant \int e^{2\psi(0, x)} \, |Du(0, x)|^2 \, dx. \tag{2.5}$$

In fact, the above inequality holds for any u from the energy space. This follows after approximating u by a sequence of more regular solutions (u_n) and passing to the limit as $n \to \infty$ in (2.5). Then, we can easily derive

$$\|Du(t, \cdot)\|_{L^2(R^n \backslash B(t^{1/2+\delta}))} = \mathcal{O}(e^{-t^{2\delta}/4}),$$

i.e. the solution decays exponentially outside every ball $B(t^{1/2+\delta})$ with $\delta > 0$. For details, see the proof of Theorem 1.3.

The following weighted energy estimates are crucial for the proof of Theorem 1.1:

PROPOSITION 2.2. *Let $u(t, x)$ be a local solution of the Cauchy problem* (1.1), (1.2) *in* $[0, T)$. *Then the following weighted energy estimate holds*: *for all* $t \in [0, T)$,

$$(t+1)^{n/4+1/2} \, \|Du(t, \cdot)\|_{L^2} \leqslant C\varepsilon + C(\max_{[0, t]} \, (\tau+1)^{\beta} \, \|e^{\delta\psi(\tau, \cdot)} u(\tau, \cdot)\|_{L^{2p}})^p. \tag{2.6}$$

Here $\psi(t, x)$ is the weight function from (2.3), $\beta > n/4p + 1/p$, *and* $\delta > 0$.

PROPOSITION 2.3. *Let $u(t, x)$ be a local solution of the Cauchy problem* (1.1), (1.2) *in* $[0, T)$. *Then the following estimate holds*: *for all* $t \in [0, T)$,

$$\|e^{\psi(t, \cdot)} \, Du(t, \cdot)\|_{L^2} \leqslant C\varepsilon + C(\max_{[0, t]} \, (\tau+1)^{\delta} \, \|e^{\gamma\psi(\tau, \cdot)} u(\tau, \cdot)\|_{L^{p+1}})^{(p+1)/2}, \tag{2.7}$$

where $\psi(t, x)$ is the weight function from (2.3), $\gamma > 2/(p+1)$, *and* $\delta > 0$.

To compare different weighted norms we need one more weighted estimate. Here we do not require $u(t, x)$ to be a solution of the Cauchy problem (1.1), (1.2). Proposition 2.4 is fulfilled for any function with compact support from the energy space.

PROPOSITION 2.4. *Let* $\theta(q)=n(1/2-1/q)$ *and* $0\leqslant\theta(q)\leqslant 1$, *and let* $0<\sigma\leqslant 1$. *If* $u\in H^1(\mathbf{R}^n)$ *with* $\operatorname{supp} u\subset B(t+K)$, $t\geqslant 0$. *Then*

$$\|e^{\sigma\psi(t,\cdot)}u\|_{L^q}\leqslant C_K(t+1)^{(1-\theta(q))/2}\,\|\nabla u\|_{L^2}^{1-\sigma}\,\|e^{\psi(t,\cdot)}\,\nabla u\|_{L^2}^{\sigma}, \tag{2.8}$$

where $\psi(t,x)$ *is the weight function from* (2.3).

We postpone the proofs of the above Propositions to the end of this section and proceed to derive the global existence Theorem 1.1 for small amplitude solutions from these weighted estimates.

Proof of the small data global existence Theorem 1.1. Let us introduce the weighted energy functional

$$W(t)=\|e^{\psi(t,\cdot)}\,Du(t,\cdot)\|_{L^2}+(1+t)^{n/4+1/2}\,\|Du(t,\cdot)\|_{L^2},$$

where the weight $(1+t)^{n/4+1/2}$ is suggested by the decay properties of the linear equation. We will show that $W(t)\leqslant C\varepsilon$ for some C depending on the initial data: this not only establishes the global existence but also shows that the solution decays at least as fast as that of the linear problem.

Adding estimates (2.6) and (2.7) from Propositions 2.2 and 2.3, we have

$$\begin{aligned}W(t)\leqslant C\varepsilon &+C(\max_{[0,t]}(\tau+1)^{\beta}\,\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}})^{p}\\ &+C(\max_{[0,t]}(\tau+1)^{\delta}\,\|e^{\gamma\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{p+1}})^{(p+1)/2}.\end{aligned}\tag{2.9}$$

We apply Proposition 2.4 to deduce

$$\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}}\leqslant C_K(\tau+1)^{(1-\theta(2p))/2-(1/2+n/4)(1-\delta)}\,W(\tau), \tag{2.10}$$

$$\|e^{\gamma\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{p+1}}\leqslant C_K(\tau+1)^{(1-\theta(p+1))/2-(1/2+n/4)(1-\gamma)}\,W(\tau), \tag{2.11}$$

where $\theta(2p)=n(1/2-1/2p)$ and $\theta(p+1)=n(1/2-1/(p+1))$. Using (2.10) and (2.11), we obtain from (2.9)

$$\begin{aligned}W(t)\leqslant C\varepsilon &+C\max_{[0,t]}(\tau+1)^{p\beta+p(1-\theta(2p))/2-p(1/2+n/4)(1-\delta)}\,(W(\tau))^{p}\\ &+C\max_{[0,t]}(\tau+1)^{(p+1)\,\delta/2+(p+1)(1-\theta(p+1))/4-(p+1)(1/2+n/4)(1-\gamma)}\\ &\times(W(\tau))^{(p+1)/2}.\end{aligned}\tag{2.12}$$

We will show that the positive constants δ, β and γ can be chosen in an appropriate way so that estimate (2.12) can be rewritten in the form

$$\max_{[0,t]}W(s)\leqslant C\varepsilon+C(\max_{[0,t]}W(s))^{(p+1)/2}+C(\max_{[0,t]}W(s))^{p}, \tag{2.13}$$

provided $p>1+2/n$. To show (2.13) we calculate the exponent of $\tau+1$ in both terms at the right side of (2.12). Letting $\beta=n/(4p)+1/p+\varepsilon_1$ and $\gamma=2/(p+1)+\varepsilon_2$, the exponent of the first term in (2.12) is

$$\begin{aligned} p[\beta+1/2-\theta(2p)/2-1/2-n/4+(1/2+n/4)\,\delta] \\ =p[\varepsilon_1+1/p+n/(2p)-n/2+(1/2+n/4)\,\delta] \\ =p[\varepsilon_1+(1/2+n/4)\,\delta]-(p-1-2/n)\,n/2, \end{aligned} \tag{2.14}$$

which is negative provided $p>1+2/n$ and ε_1 and δ are chosen small enough.

For the exponent of the second term on the right side of (2.12) we have

$$\begin{aligned} (p+1)[\delta/2+1/4-\theta(p+1)/4-1/2-n/4+(1/2+n/4)\,\gamma] \\ \leqslant (p+1)[\varepsilon_2/2+1/(p+1)-1/4-n/4+(1/2+n/4)\,\gamma] \\ =(p+1)[\varepsilon_2/2+(1/2+n/4)\,\gamma]-[(p+1)(n+1)/4-1], \end{aligned} \tag{2.15}$$

which is negative provided $p>1$ and ε_2 and δ are small enough. Then, using (2.14) and (2.15), we can rewrite (2.12) like (2.13). Denote now $M(t)=\max_{[0,\,t]} W(s)$: from (2.13) we then have $M(t)\leqslant C\varepsilon$, for ε small enough, i.e.

$$W(t)=\|e^{\psi(t,\,\cdot)}\,Du(t,\cdot)\|_{L^2}+(1+t)^{n/4+1/2}\,\|Du(t,\cdot)\|_{L^2}\leqslant C\varepsilon. \tag{2.16}$$

This completes the proof of the small data global existence Theorem 1.1.

Proof of Theorem 1.3. From estimate (2.16) and inequality (2.4), satisfied by $\psi(t,x)$, we deduce

$$\begin{aligned} C\varepsilon &\geqslant \|e^{\psi(t,\,\cdot)}\,Du(t,\cdot)\|_{L^2(R^n)} \\ &\geqslant \|e^{|\cdot|^2/4(t+K)}\,Du(t,\cdot)\|_{L^2(R^n\setminus B(t^{1/2+\delta}))} \\ &\geqslant e^{t^{1+2\delta}/4(t+K)}\,\|Du(t,\cdot)\|_{L^2(R^n\setminus B(t^{1/2+\delta}))}. \end{aligned}$$

This inequality shows that the solution of the Cauchy problem (1.1), (1.2) decays exponentially ouside every ball $B(t^{1/2+\delta})$, $\delta>0$.

We now prove the weighted estimates of Propositions 2.2–2.4. The proof of Proposition 2.2 uses a decay estimate for the dissipative wave equation due to Matsumura [13]. To state this estimate, we let $\delta_0\in\mathscr{D}'(\mathbf{R}^{1+n})$ be the Dirac delta distribution supported at 0 and $S_2\in\mathscr{D}'(\mathbf{R}\times\mathbf{R}^n)$ be the solution of $(\Box+\partial_t)\,S_2=\delta_0$ supported in the forward light cone

$$C_+=\{(t,x)\in\mathbf{R}_+\times\mathbf{R}^n:|x|<t\}. \tag{2.17}$$

We denote $S_2(t) * f(x) = \int_{R^n} S_2(t, x-y) f(y)\, dy$, for a suitable function f. We can now state Matsumura's result as follows:

PROPOSITION 2.5. *Let $m \in [1, 2]$. Then*

$$\|\partial_t^k \nabla_x^\alpha S_2(t) * f\|_{L^2} \leqslant C(1+t)^{n/4 - n/(2m) - |\alpha|/2 - k} (\|f\|_{L^m} + \|f\|_{H^{k+|\alpha|-1}}),$$

for each $f \in L^m(\mathbf{R}^n) \cap H^{k+|\alpha|-1}(\mathbf{R}^n)$.

Proof of Proposition 2.2. We begin by rewriting the Cauchy problem (1.1), (1.2) as an integral equation,

$$u(t) = \varepsilon u_L(t) + \int_0^t S_2(t-\tau) * |u(\tau)|^p\, d\tau, \tag{2.18}$$

where $u_L(t) = \partial_t S_2(t) * u_0 + S_2(t) * (u_0 + u_1)$ is the solution of the linear equation $(\square + \partial_t)\, u_L = 0$, with data $u_L|_{t=0} = u_0$, $\partial_t u_L|_{t=0} = u_1$. We estimate the norm $\|Du(t, \cdot)\|_{L^2}$ by means of Proposition 2.5. The linear term $\|Du_L(t, \cdot)\|_{L^2}$ is bounded by

$$\begin{aligned} \|Du_L(t, \cdot)\|_{L^2} &\leqslant C\varepsilon(t+1)^{-n/4-1/2} (\|u_0\|_{H^1} + \|u_0\|_{L^1} + \|u_1\|_{L^2} + \|u_1\|_{L^1}) \\ &\leqslant C\varepsilon(t+1)^{-n/4-1/2}. \end{aligned} \tag{2.19}$$

To estimate the nonlinear term in (2.18), we split the integral into the two parts

$$\int_0^{t/2} S_2(t-\tau) * |u(\tau)|^p\, d\tau + \int_{t/2}^t S_2(t-\tau) * |u(\tau)|^p\, d\tau. \tag{2.20}$$

For the first integral we apply Proposition 2.5, with $m = 1$, and obtain

$$\|DS_2(t-\tau) * |u(\tau)|^p\|_{L^2} \leqslant C(t-\tau+1)^{-n/4-1/2} (\|u(\tau)\|_{L^p}^p + \|u(\tau)\|_{L^{2p}}^p). \tag{2.21}$$

To transform the L^p norm into a weighted L^{2p} norm, we use the Cauchy inequality

$$\begin{aligned} \|u(\tau, \cdot)\|_{L^p}^p &\equiv \int_{B(\tau+K)} |u(\tau, x)|^p\, dx \\ &\leqslant \left(\int_{B(\tau+K)} e^{-2p\,\delta\psi(\tau, x)}\, dx \right)^{1/2} \left(\int_{B(\tau+K)} e^{2p\,\delta\psi(\tau, x)} |u(\tau, x)|^{2p}\, dx \right)^{1/2}, \end{aligned}$$

where $\delta > 0$. Since $\psi(\tau, x) \geqslant |x|^2/4(t+K)$ for $x \in B(\tau+K)$, the first integral is bounded by

$$\int_{B(\tau+K)} e^{-p\delta\,|x|^2/2(\tau+K)}\,dx \leqslant \int_{R^n} e^{-p\delta\,|x|^2/2(\tau+K)}\,dx \equiv \left(\frac{2\pi}{p\delta}\right)^{n/2} (\tau+K)^{n/2}.$$

Thus, for the norm $\|u(\tau,\cdot)\|_{L^p}$ we obtain the weighted estimate

$$\|u(\tau,\cdot)\|_{L^p}^p \leqslant C_{K,\delta}(\tau+1)^{n/4}\,\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}}^p, \tag{2.22}$$

where $\delta > 0$. Since also $\psi > 0$, the estimate

$$\|u(\tau,\ )\|_{L^{2p}}^p \leqslant C_\delta(\tau+1)^{n/4}\,\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}}^p \tag{2.23}$$

obviously holds. Therefore, inserting estimates (2.22) and (2.23) into (2.21) we have

$$\begin{aligned}
&\int_0^{t/2} \|DS_2(t-\tau) * |u(\tau,\cdot)|^p\|_{L^2}\,d\tau \\
&\quad\leqslant C\int_0^{t/2} (t-\tau+1)^{-n/4-1/2}\,((\tau+1)^{n/(4p)}\,\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}})^p\,d\tau \\
&\quad\leqslant C(t+1)^{-n/4-1/2}\,(\max_{[0,t]}\,(\tau+1)^\beta\,\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}})^p,
\end{aligned} \tag{2.24}$$

for any $\beta > n/4p + 1/p$.

To estimate the second integral of (2.20) we apply Proposition 2.5 with $m = 2$ and obtain

$$\|DS_2(t-\tau) * |u(\tau)|^p\|_{L^2} \leqslant C(t-\tau+1)^{-1/2}\,\|u(\tau)\|_{L^{2p}}^p.$$

From the inequality

$$\|u(\tau,\cdot)\|_{L^{2p}}^p \leqslant C(t+1)^{-n/4-1}\,(\max_{[t/2,t]}\,(\tau+1)^{n/(4p)+1/p}\,\|u(\tau,\cdot)\|_{L^{2p}})^p,$$

it follows that

$$\begin{aligned}
&\int_{t/2}^{t} \|DS_2(t-\tau) * |u(\tau,\cdot)|^p\|_{L^2}\,d\tau \\
&\quad\leqslant C\int_{t/2}^{t} (t-\tau+1)^{-1/2}\,\|u(\tau,\cdot)\|_{L^{2p}}^p\,d\tau \\
&\quad\leqslant C(t+1)^{-n/4-1/2}\,(\max_{[t/2,t]}\,(\tau+1)^{n/(4p)+1/p}\,\|e^{\delta\psi(\tau,\cdot)}u(\tau,\cdot)\|_{L^{2p}})^p,
\end{aligned} \tag{2.25}$$

where in the last step we have used the inequalities $\int_{t/2}^{t} (t-\tau+1)^{-1/2}\, d\tau \leqslant C(t+1)^{1/2}$ and $\|u(\tau,\cdot)\|_{L^{2p}}^{p} \leqslant \|e^{\delta\psi(\tau,\cdot)} u(\tau,\cdot)\|_{L^{2p}}^{p}$ for any $\delta > 0$. Combining (2.24), (2.25), and (2.19), we complete the proof of Proposition 2.2.

Proof of Proposition 2.3. Let us assume, for the moment, that $u \in C^2([0, T), L^2(\mathbf{R}^n))$. We multiply Eq. (1.1) by $e^{2\psi} u_t$ and use the identity (2.1) to obtain

$$\begin{aligned} -\frac{2\psi_t}{p+1} e^{2\psi} |u|^p u = {} & \frac{d}{dt}\left(\frac{e^{2\psi}}{2}(|u_t|^2 + |\nabla u|^2) - \frac{e^{2\psi}}{p+1} |u|^p u\right) \\ & - \operatorname{div}(e^{2\psi} u_t \nabla u) \\ & - \frac{e^{2\psi}}{\psi_t} (\psi_t \nabla u - u_t \nabla \psi)^2. \end{aligned}$$

Integrating over the strip $[0, t] \times \mathbf{R}^n$ and recalling that $\psi_t < 0$, we have

$$\begin{aligned} \|e^{\psi(t,\cdot)} Du(t,\cdot)\|_{L^2}^2 \leqslant {} & C\varepsilon + C\, \|e^{2/(p+1)\, \psi(t,\cdot)} u(t,\cdot)\|_{L^{p+1}}^{p+1} \\ & + C \int_0^t \Big(\max_{\operatorname{supp} u(\tau,\cdot)} \phi(\tau, x)\Big)\, \|e^{\gamma\psi(\tau,\cdot)} u(\tau,\cdot)\|_{L^{p+1}}^{p+1}\, d\tau, \end{aligned} \tag{2.26}$$

where

$$\phi(\tau, x) = |\psi_\tau(\tau, x)|\, e^{(2-\gamma(p+1))\, \psi(\tau, x)}, \qquad \gamma > 2/(p+1).$$

Actually, estimate (2.26) holds for any solution u from the energy space. This can be established by a suitable approximation argument.

To complete the proof, we show that if $\gamma > 2/(p+1)$ then

$$\max_{\operatorname{supp} u(\tau,\cdot)} \phi(\tau, x) \leqslant \frac{C}{\tau+1}. \tag{2.27}$$

Note that

$$\begin{aligned} |\psi_\tau(\tau, x)| &= \frac{1}{2}\left|1 - \frac{\tau+K}{[(\tau+K)^2 - |x|^2]^{1/2}}\right| \\ &\leqslant \frac{C\,|x|^2}{(\tau+K)[(\tau+K)^2 - |x|^2]^{1/2}}. \end{aligned}$$

Thus, for $|x| \leqslant (\tau+K)/2$, we have $|\psi_\tau(\tau, x)| \leqslant C\,|x|^2/(\tau+K)^2$. Moreover, we know from (2.4) that $\psi(\tau, x) \geqslant |x|^2/[4(\tau+K)]$. Therefore,

$$\begin{aligned}\phi(\tau, x) &\leqslant \frac{C}{\tau+K} \cdot \frac{|x|^2}{4(\tau+K)}\, e^{(2-\gamma(p+1))(|x|^2/4(\tau+K))}\\ &\leqslant \frac{C_K}{\tau+1},\end{aligned} \tag{2.28}$$

where in the above estimate we used the fact that $\sup_{r \geqslant 0} re^{(2-\gamma(p+1))\,r}$ is finite for $\gamma > 2/(p+1)$.

For $|x| > (s+K)/2$ we have $\psi(\tau, x) \geqslant (\tau+K)/16$ and $|\psi_\tau(\tau, x)| \leqslant C(\tau+K)/d$, where $d > 0$ is the distance between $\operatorname{supp} u_0 \cup \operatorname{supp} u_1$ and $\mathbf{R}^n \backslash B(K)$. Since $\sup_{r \geqslant 0} r^2 e^{(2-\gamma(p+1))\,r}$ is finite for $\gamma > 2/(p+1)$,

$$\begin{aligned}\phi(\tau, x) &\leqslant \frac{C_d}{\tau+K} \cdot (\tau+K)^2\, e^{(2-\gamma(p+1))((\tau+K)/16)}\\ &\leqslant \frac{C_{K,d}}{\tau+1}.\end{aligned} \tag{2.29}$$

Applying estimates (2.28) and (2.29), we obtain (2.27) and complete the proof of the Proposition 2.3.

To prove the weighted Sobolev estimate of Proposition 2.4 we need the following Lemma:

Lemma 2.1. *Let* $u \in H^1(\mathbf{R}^n)$ *and* $\operatorname{supp} u \subset B(t+K)$, $t \geqslant 0$. *Then, for* $\sigma > 0$, *we have*

$$\frac{\sigma n}{2}(t+K)^{-1}\, \|e^{\sigma\psi(t,\cdot)} u\|_{L^2}^2 + \|\nabla(e^{\sigma\psi(t,\cdot)} u)\|_{L^2}^2 \leqslant \|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2}^2,$$

where $\psi(t, x)$ *is the weight function* (2.3).

Proof. We let $f = e^{\sigma\psi} u$, $\sigma > 0$, and compute

$$\nabla u \equiv \nabla(e^{-\sigma\psi} f) = e^{-\sigma\psi}(\nabla f - \sigma f\, \nabla\psi).$$

Thus, $e^{\sigma\psi}\, \nabla u = \nabla f - \sigma f\, \nabla\psi$ and

$$\begin{aligned}\|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2}^2 = &\int |\nabla f|^2\, dx + \sigma^2 \int f^2\, |\nabla\psi|^2\, dx\\ &- 2\sigma \int f(\nabla f \cdot \nabla\psi)\, dx.\end{aligned} \tag{2.30}$$

Since $f\nabla f = \nabla f^2/2$, integrating by parts transforms the third term into

$$\sigma \int (\Delta\psi)\, f^2\, dx.$$

Differentiating (2.3) twice, we obtain

$$\Delta\psi(t, x) = \frac{n}{2}\,((t+K)^2 - |x|^2)^{-1/2} + \frac{|x|^2}{2}\,((t+K)^2 - |x|^2)^{-3/2};$$

therefore, $\Delta\psi(t, x) \geqslant \frac{n}{2}(t+K)^{-1}$. From (2.30), it follows that

$$\|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2}^2 \geqslant \|\nabla f\|_{L^2}^2 + \frac{\sigma n}{2}\,(t+K)^{-1}\, \|f\|_{L^2}^2,$$

which completes the proof of Lemma 2.1.

Proof of Proposition 2.4. Applying the Gagliardo–Nirenberg inequality to $f = e^{\sigma\psi} u$, we have

$$\|f\|_{L^q} \leqslant C\, \|f\|_{L^2}^{1-\theta(q)}\, \|\nabla f\|_{L^2}^{\theta(q)},$$

where $\theta(q) = n(1/2 - 1/q)$ is such that $0 \leqslant \theta(q) \leqslant 1$. By Lemma 2.1,

$$\|\nabla f\|_{L^2} \leqslant \|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2},$$

$$\|f\|_{L^2} \leqslant \left(\frac{2}{\sigma n}\right)^{1/2} (t+K)^{1/2}\, \|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2},$$

so that

$$\|f\|_{L^q} \leqslant C(t+K)^{(1-\theta(q))/2}\, \|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2}. \tag{2.31}$$

To complete the proof, we combine (2.31) with the interpolation estimate

$$\|e^{\sigma\psi(t,\cdot)}\, \nabla u\|_{L^2} \leqslant \|\nabla u\|_{L^2}^{1-\sigma}\, \|e^{\psi(t,\cdot)}\, \nabla u\|_{L^2}^{\sigma},$$

$\sigma \in (0, 1]$, for the proof of which we write $e^{2\sigma\psi}\, |\nabla u|^2 = e^{2\sigma\psi}\, |\nabla u|^{2\sigma} \cdot |\nabla u|^{2(1-\sigma)}$ and apply Hölder's inequality with conjugate exponents $1/\sigma,\ 1/(1-\sigma)$.

3. BLOW-UP

The proof of blow-up Theorem 1.2 is split into two parts. In the first, the result is established for exponents in the smaller range $1 < p < 1 + 1/n$, which can be done relatively easily. For the proof, we derive a nonlinear differential inequality satisfied by the average

$$F(t) = \int u(t, x)\, dx, \tag{3.1}$$

where u is the local solution of (1.1), (1.2). Note that the support of $u(t, \cdot)$ is a compact set and, therefore, the integral exists as long as the solution exists. It can also be shown, using the equation and standard approximation arguments, that $F(t)$ is twice continuously differentiable with respect to t.

We will show that F blows up in finite time by resorting to a blow-up result for ordinary differential inequalities. Thus, the solution u must also blow up in finite time.

Proof of Theorem 1.2 *for* $1 < p < 1 + 1/n$. Integrating Eq. (1.1) over the whole space and applying the divergence theorem to Δu, we obtain

$$\ddot{F}(t) + \dot{F}(t) = \|u(t, \cdot)\|_{L^p}^p. \tag{3.2}$$

In addition, an upper estimate for F follows from the Hölder inequality and the finite propagation speed property, i.e. $\operatorname{supp} u(t, \cdot) \subset B(t+K)$:

$$\begin{aligned} |F(t)| &\leqslant \left(\int_{B(t+K)} dx \right)^{(p-1)/p} \left(\int |u(t, x)|^p\, dx \right)^{1/p} \\ &\leqslant C(t+K)^{n(p-1)/p}\, \|u(t)\|_{L^p}. \end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3) we deduce the basic differential inequality.

$$\ddot{F}(t) + \dot{F}(t) \geqslant C(t+K)^{-n(p-1)}\, |F(t)|^p. \tag{3.4}$$

That F is actually forced to blow up is a consequence of the following result:

PROPOSITION 3.1. *Let* $0 \geqslant A > -1$ *and* $r > 0$. *Assume that* $F(t)$ *is a twice continuously differentiable solution of the inequality*

$$\ddot{F}(t) + \dot{F}(t) \geqslant C_0(t+K)^A\, |F(t)|^{1+r}, \qquad t > 0, \tag{3.5}$$

with $C_0 > 0$, *such that* $F(0) > 0$ *and* $\dot{F}(0) > 0$. *Then* $F(t)$ *blows up in finite time.*

Note that $F(0) = \varepsilon c_0$ and $\dot{F}(0) = \varepsilon c_1$, for F defined by (3.1), and c_0 and c_1 are positive by the assumptions of Theorem 1.2; thus, applying

Proposition 3.1 to (3.4), we conclude that F blows up in finite time if $1<p<1+1/n$. So, to complete this case we will prove this proposition.

Proof of Proposition 3.1. To show that the ordinary differential inequality (3.5) has no global solutions, we consider the auxiliary initial value problem

$$\dot{Y}(t)=v(t+K)^{A}\,[\,Y(t)]^{1+r/2},\qquad Y(0)\equiv F(0)>0, \tag{3.6}$$

where $v>0$ is a small number to be chosen later. Since

$$Y(t)=\left([\,Y(0)]^{-r/2}-\frac{vr}{2(A+1)}\,[(t+K)^{A+1}-K^{A+1}]\right)^{-2/r},$$

and $A>-1$, the solution $Y(t)$ of the above problem blows up at a finite time T_0 and satisfies $Y(t)>Y(0)>0$ for $0\leqslant t<T_0$. We compute

$$\begin{aligned}\ddot{Y}(t)&=v(1+r/2)(t+K)^{A}\,[\,Y(t)]^{r/2}\,\dot{Y}(t)+vA(t+K)^{A-1}\,[\,Y(t)]^{1+r/2}\\&\leqslant v^2(1+r/2)(t+K)^{2A}\,[\,Y(t)]^{1+r},\end{aligned}\tag{3.7}$$

where we have used that $A\leqslant 0$ and that Y satisfies Eq. (3.6). Adding (3.6) and (3.7), and observing that $2A\leqslant A$ and $[\,Y(t)]^{1+r/2}<[\,Y(0)]^{-r/2}$ $[\,Y(t)]^{1+r}$, we have

$$\begin{aligned}\ddot{Y}(t)+\dot{Y}(t)&\leqslant v^2(1+r/2)(t+K)^{2A}\,[\,Y(t)]^{1+r}+v(t+K)^{A}\,[\,Y(t)]^{1+r/2}\\&\leqslant B(t+K)^{A}\,[\,Y(t)]^{1+r},\end{aligned}$$

where $B=v^2(1+r/2)+vA[\,Y(0)]^{-r/2}$.

Further, we choose v so small that

$$B=v^2(1+r/2)+vA[\,Y(0)]^{-r/2}<C_0,$$
$$\dot{Y}(0)=vK^{A}[\,Y(0)]^{1+r/2}<\dot{F}(0).$$

Then we obtain the inequality

$$\ddot{Y}(t)+\dot{Y}(t)\leqslant C_0(t+K)^{A}\,[\,Y(t)]^{1+r} \tag{3.8}$$

and the initial conditions $Y(0)\leqslant F(0)$ and $\dot{Y}(0)<\dot{F}(0)$.

We can now show that $F(t)\geqslant Y(t)$ for $0\leqslant t<T_0$, so that $F(t)$ also blows up in finite time. From $\dot{F}(0)>\dot{Y}(0)$ we have $\dot{F}(t)>\dot{Y}(t)$ for t small enough; so we set

$$t_0=\sup\{t\in[0,T_0)\mid \dot{F}(\tau)>\dot{Y}(\tau)\text{ for }0\leqslant\tau<t\}.$$

Suppose $t_0 < T_0$, where T_0 is the blow up time for $Y(t)$: thus, $\dot{F}(t) > \dot{Y}(t)$ for $t \in [0, t_0)$ and $\dot{F}(t_0) = \dot{Y}(t_0)$. Since $\dot{F}(t) - \dot{Y}(t) > 0$, the function $F(t) - Y(t)$ is strictly increasing in the interval $0 \leqslant t < t_0$. In particular $F(t) - Y(t) > F(0) - Y(0) = 0$ for such t. We note that $F(t_0) > Y(t_0)$, because if $F(t_0) = Y(t_0)$ then the function $F(t) - Y(t)$ would have zeros at 0 and at t_0, so its derivative would vanish between 0 and t_0, i.e. $\dot{F}(t_1) = \dot{Y}(t_1)$ for some $0 < t_1 < t_0$, which is impossible by the definition of t_0. Therefore, $F(t_0) > Y(t_0)$ and $\dot{F}(t_0) = \dot{Y}(t_0)$.

On the other hand, subtracting (3.5) from (3.8) we have

$$[\ddot{F}(t) - \ddot{Y}(t)] + [\dot{F}(t) - \dot{Y}(t)] \geqslant C(t+K)^{2A} \{[F(t)]^{1+r} - [Y(t)]^{1+r}\}$$
$$\geqslant 0,$$

for $0 \leqslant t \leqslant t_0$. We can rewrite the above inequality in the form

$$\frac{d}{dt} e^{t}[\dot{F}(t) - \dot{Y}(t)] \geqslant 0,$$

and integrate over $[0, t_0]$ to obtain

$$e^{t_0}[\dot{F}(t_0) - \dot{Y}(t_0)] \geqslant \dot{F}(0) - \dot{Y}(0),$$

i.e. $\dot{F}(t_0) - \dot{Y}(t_0) > 0$. We come to a contradiction: thus, $t_0 \geqslant T_0$, and the proof of Proposition 3.1 is complete.

Proof of Theorem 1.2 *for* $1 + 1/n \leqslant p < 1 + 2/n$. The preceding argument is not sufficient, since $p(n-1) \geqslant 1$ and Proposition 3.1 does not apply. We can obtain an inequality stronger than (3.4), i.e. with an exponent less than $n(p-1)$, if we could integrate over a smaller ball in (3.3); so, it is important to know where the support of the solution is actually contained. In the global existence case, $p > 1 + 2/n$, we have established that the solution of (1.1), (1.2) has fast (exponential) decay outside any region of the form $|x| < t^{1/2+\delta}$, with $\delta > 0$; the conjecture that this always happens, regardless of the value of p, seems plausible, although we are not able to prove it.

Instead, we achieve a crucial improvement of (3.4) by showing that F satisfies the stronger lower estimate given below. Our approach to derive such an estimate is to consider a space-time average of the solution u which is adjusted to the dissipative wave operator $\square + \partial_t$. The role is played by the convolution $S_{2k} * Hu$, where $S_{2k} \in \mathscr{D}'(\mathbf{R}^{1+n})$ is the fundamental solution of $(\square + \partial_t)^k$ supported in C_+, the forward light cone (2.17). We will use the fundamental solutions S_{2k} with $k \geqslant (n+1)/2$, taking advantage of the fact that, in this case, S_{2k} are positive continuous functions.

The lower estimate for F is given by

PROPOSITION 3.2. *Let the assumptions of Theorem* 1.2 *hold, i.e.* $1 < p < 1 + 2/n$ *and*

$$c_i \equiv \int u_i(x)\, dx > 0, \qquad i = 0, 1.$$

Then for each $B \geqslant 0$ *there exists* $C_B > 0$ *such that*

$$F(t) \geqslant C_B (t+K)^B, \qquad t \geqslant 0. \tag{3.9}$$

We can now derive a stronger version of (3.4) by writing $|F|^p$ as $|F|^{(p-1)/2}\, |F|^{(p+1)/2}$ and using Proposition 3.2: we come up with the modified inequality

$$\ddot{F}(t) + \dot{F}(t) \geqslant C(t+K)^{(p-1)(B/2-n)}\, |F(t)|^{(p+1)/2}. \tag{3.10}$$

Choosing $B = n$ and applying Proposition 3.2 we complete the proof of Theorem 1.2.

Thus, it remains to prove Proposition 3.2. Let us recall that the idea is to consider the average

$$\begin{aligned}[S_{2k} * Hu](t, x) &\equiv \int_{R^{n+1}} S_{2k}(t-s, x-y)\, H(s)\, u(s, y)\, ds\, dy \\ &= \int_{[0,t] \times R^n} S_{2k}(t-s, x-y)\, u(s, y)\, ds\, dy,\end{aligned}$$

and to obtain the lower bound (3.9) on F from a suitable lower bound on $S_{2k} * Hu$. We have to use some properties of S_{2k}, which are well known, and to establish some asymptotic estimates for integrals involving modified Bessel functions. The expression and some other important properties of S_{2k} are collected in the following lemma.

LEMMA 3.1. *Let* S_{2k} *be given by*

$$S_{2k}(t, x) = \begin{cases} \dfrac{e^{-t/2}\left(\sqrt{t^2 - |x|^2}\right)^{k-(n+1)/2} I_{k-(n+1)/2}\left(\frac{1}{2}\sqrt{t^2 - |x|^2}\right)}{\pi^{(n-1)/2} 2^n \Gamma(k)}, & (t, x) \in C_+, \\ 0, & (t, x) \notin C_+, \end{cases} \tag{3.11}$$

where Γ and $I_{k-(n+1)/2}$ are respectively the Gamma function and the modified Bessel function of order $k-(n+1)/2$.

Then S_{2k} is an entire function of k, valued in $\mathscr{D}'(\bar{C}_+)$, which satisfies

$$\begin{aligned} S_{2k} * S_{2l} &= S_{2(k+l)}, \qquad \forall k, l \in \mathbf{C}, \\ (\Box + \partial_t)\, S_{2k} &= S_{2(k-1)}, \qquad \text{and} \qquad S_0 = \delta_0. \end{aligned} \tag{3.12}$$

Here $\delta_0 \in \mathscr{D}'(\mathbf{R}^{n+1})$ is the Dirac delta distribution supported at 0.

Moreover, $S_{2k} \in C(\mathbf{R}_+ \times \mathbf{R}^n)$ and $S_{2k} \geqslant 0$ for $k \geqslant (n+1)/2$.

(For an outline of the proof, see the Appendix.)

To demonstrate the relationship between F and $S_{2k} * Hu$, we turn to (3.2): in fact, we can treat (3.2) as an ordinary differential equation for $F(t)$, whose solution is given by

$$F(t) = \varepsilon c_0 + \varepsilon c_1 (1 - e^{-t}) + \int_0^t (1 - e^{s-t}) \left(\int |u(s, x)|^p \, dx \right) ds, \tag{3.13}$$

where $c_0 = \int u_0 \, dx$ and $c_1 = \int u_1 \, dx$ are positive by assumption. From (3.13) we immediately deduce the lower bound

$$\begin{aligned} F(t) &\geqslant (1 - e^{-1}) \int_0^{t-1} \int |u(s, x)|^p \, dx \, ds, \qquad t \geqslant 1 \\ &= (1 - e^{-1}) \, \|u\|^p_{L^p([0, t-1] \times R^n)}. \end{aligned} \tag{3.14}$$

Further, the norm $\|u\|_{L^p([0, t-1] \times R^n)}$ can be related to $S_{n+1} * Hu$ by Hölder's inequality: For each $t \geqslant 1$ and $x \in \mathbf{R}^n$,

$$\begin{aligned} |[S_{n+1} * Hu](t-1, x)| &\leqslant \int_{[0, t-1] \times R^n} S_{n+1}(t-1-s, x-y) \, |u(s, y)| \, ds \, dy \\ &\leqslant \|S_{n+1}\|_{L^{p'}([0, t-1] \times R^n)} \, \|u\|_{L^p([0, t-1] \times R^n)}, \end{aligned}$$

with $p' = p/(p-1)$. Therefore, we have

$$\|u\|^p_{L^p([0, t-1] \times R^n)} \geqslant |S_{n+1} * Hu(t-1, x)|^p / \|S_{n+1}\|^p_{L^{p'}([0, t-1] \times R^n)}.$$

Combining this lower estimate with (3.13) and (3.14), we obtain

$$F(t) \geqslant \frac{|[S_{n+1} * Hu](t-1, x)|^p}{2 \, \|S_{n+1}\|^p_{L^{p'}([0, t-1] \times R^n)}}, \qquad t \geqslant 1, \quad x \in \mathbf{R}^n, \tag{3.15}$$

$$F(t) \geqslant \varepsilon c_0, \qquad t \geqslant 0. \tag{3.16}$$

Having found the above, we can deduce Proposition 3.2 from a lower bound on $[S_{n+1} * Hu](t, x)$ and an upper bound on $\|S_{n+1}\|_{L^{p'}([0,\, t-1] \times R^n)}$. Formula (3.11) and the asymptotics of the modified Bessel function of order v (cf. [13], Chapter 7), namely

$$I_v(\rho/2) = \frac{e^{\rho/2}}{\sqrt{\pi\rho}} \left(1 + \mathcal{O}\left(\frac{1}{\rho}\right)\right), \qquad \rho \to \infty, \tag{3.17}$$

suggest that the strongest lower estimate for large t is obtained if $|x|$ is small, say $|x| \leqslant 1$. We state the results we need as a sequence of lemmas, which we prove at the end of this section.

To study the asymptotic behavior of $[S_{2k} * Hu](t, x)$, we derive an integral identity from (1.1) and (1.2).

LEMMA 3.2. *Let u be a local solution of* (1.1), (1.2). *For $k \geqslant (n+3)/2$, we have*

$$S_{2(k-1)} * Hu = \varepsilon \partial_t S_{2k}(t) * u_0 + \varepsilon S_{2k}(t) * (u_0 + u_1) + S_{2k} * H\,|u|^p, \tag{3.18}$$

where $S_{2k}(t) * f(x)$ denotes the spatial convolution, i.e.

$$S_{2k}(t) * f(x) = \int_{R^n} S_{2k}(t, x-y)\, f(y)\, dy.$$

The asymptotics of the linear term in (3.18) is given next.

LEMMA 3.3. *Let $k \geqslant (n+3)/2$ be fixed. For $|x| \leqslant 1$, we have the following asymptotics as $t \to \infty$:*

$$S_{2k}(t) * (u_0 + u_1) = \frac{c_0 + c_1}{2^n \pi^{n/2} \Gamma(k)}\, t^{k-(n+2)/2} + \mathcal{O}(t^{k-(n+4)/2}), \tag{3.19}$$

$$\partial_t S_{2k}(t) * u_0 = \mathcal{O}(t^{k-(n+4)/2}). \tag{3.20}$$

Let us remind that $c_i \equiv \int u_i(x)\, dx > 0$, $i = 0, 1$, due to the assumptions of Theorem 1.2.

The last result is an upper bound on the L^q norm of S_{2k}, which appears in the denominators of some estimates.

LEMMA 3.4. *Let $k \geqslant (n+1)/2$ and $1 \leqslant q < \infty$. The following estimate holds,*

$$\|S_{2k}\|_{L^q([0,\, t] \times R^n)} \leqslant C t^{k + 1/q - n/(2q') - 1}, \qquad q' = q/(q-1),$$

for each $t \geqslant 1$.

We postpone the proofs of these lemmas and proceed to apply them to prove Proposition 3.2.

Proof of Proposition 3.2. In view of (3.15), (3.16), we can derive this proposition from an appropriate lower estimate of $[S_{n+1} * Hu](t, x)$ for $|x| \leqslant 1$ and large t. First, we derive a weak lower estimate by Hölder's inequality, and then we strenghten it by m iterations. We choose the integer m so that we assure sufficiently fast growth of $S_{n+1} * Hu$. For instance, we can choose m satisfying

$$[1/(p-1) - n/2]\, p^m - (p+1)/(p-1) \geqslant B/p. \tag{3.21}$$

This inequality holds, for large m, since $1/(p-1) - n/2 > 0$ and $p > 1$. The above choice of m is justified at the end of the proof.

By Lemma 3.3, there exists $t_m > 1$ such that the linear term in (3.18) satisfies

$$\varepsilon \frac{\partial}{\partial t} S_{2k}(t) * u_0 + \varepsilon S_{2k}(t) * (u_0 + u_1) \geqslant C_m \varepsilon t^{k-(n+2)/2}, \qquad t \geqslant t_m, \qquad |x| \leqslant 1, \tag{3.22}$$

with $C_m > 0$, for all k in the interval $(n+3)/2 \leqslant k \leqslant (n+3)/2 + m$. The positivity of the linear term, Lemma 3.2, and the fact that $S_{2k} \geqslant 0$, for $k \geqslant (n+1)/2$ imply $0 \leqslant S_{2k} * Hu$. Moreover

$$S_{2k} * H\, |u|^p \leqslant S_{2(k-1)} * Hu,$$

for $(n+3)/2 \leqslant k \leqslant (n+1)/2 + m$. Applying Hölder's inequality to $S_{2k} * Hu$, we have

$$\begin{aligned} |S_{2k} * Hu|^p &\leqslant \left(\int_{R^{n+1}} [S_{2k}(t-s, x-y)]^{(p-1)/p + 1/p} H(s)\, |u(s, y)|\, ds\, dy \right)^p \\ &\leqslant \|S_{2k}\|^{p-1}_{L^1([0,t] \times R^n)} (S_{2k} * H\, |u|^p). \end{aligned}$$

From the last two estimates, we deduce

$$S_{2(k-1)} * Hu \geqslant (S_{2k} * Hu)^p / \|S_{2k}\|^{p-1}_{L^1([0,t] \times R^n)}, \tag{3.23}$$

where $(n+3)/2 \leqslant k \leqslant (n+1)/2 + m$. This is the lower estimate, which we iterated m times, yielding the following chain of lower bounds:

$$\begin{aligned}S_{n+1} * Hu &\geqslant (S_{n+3} * Hu)^p / \|S_{n+3}\|^{p-1}_{L^1([0,t]\times R^n)}\\&\geqslant (S_{n+5} * Hu)^{p^2} / \|S_{n+3}\|^{p-1}_{L^1([0,t]\times R^n)} \, \|S_{n+5}\|^{(p-1)p}_{L^1([0,t]\times R^n)})\\&\geqslant \cdots\\&\geqslant (S_{n+2m+1} * Hu)^{p^m} \Big/ \prod_{l=1}^{m} (\|S_{n+2l+1}\|_{L^1([0,t]\times R^n)})^{(p-1)p^{l-1}}.\end{aligned}$$

By $S_{n+2m+3} \geqslant 0$ and identity we obtain from Lemma 3.2,

$$S_{n+2m+1} * Hu \geqslant \varepsilon \frac{\partial}{\partial t} S_{n+2m+3}(t) * u_0 + \varepsilon S_{n+2m+3}(t) * (u_0 + u_1).$$

Therefore, the chain of estimates gives

$$S_{n+1} * Hu \geqslant \frac{\left(\varepsilon \frac{\partial}{\partial t} S_{n+2m+3}(t) * u_0 + \varepsilon S_{n+2m+3}(t) * (u_0+u_1)\right)^{p^m}}{\prod_{l=1}^{m} (\|S_{n+2l+1}\|_{L^1([0,t]\times R^n)})^{(p-1)p^{l-1}}}.$$

Estimating the numerator by (3.22), with $k = (n+3)/2 + m$, and the denominator by Lemma 3.4, with $k = (n+1)/2 + l$ and $l = 1, ..., m$, we have

$$\begin{aligned}S_{n+1} * Hu &\geqslant (C_m \varepsilon t^{m+1/2})^{p^m} \Big/ \prod_{l=1}^{m} (C t^{l+(n+1)/2})^{(p-1)p^{l-1}}\\&\geqslant C_{m,\varepsilon} t^{d(p,m)}.\end{aligned} \tag{3.24}$$

Here $C_{m,\varepsilon} > 0$ and $d(p,m) = (m+1/2)\, p^m - \sum_{l=1}^{m} (l + (n+1)/2)(p-1)\, p^{l-1}$. It is easy to verify the identities

$$(p-1) \sum_{l=1}^{m} p^{l-1} = p^m - 1,$$

$$(p-1) \sum_{l=1}^{m} l p^{l-1} = (m - 1/(p-1))\, p^m + 1/(p-1);$$

thus, we have

$$d(p,m) = [1/(p-1) - n/2]\, p^m + (n-1)/2 - 1/(p-1). \tag{3.25}$$

We will show that $d(p,m)$ is large enough to assure the lower estimate of Proposition 3.2. We turn back to (3.15). Applying Lemma 3.4 to obtain an upper bound for the denominator,

$$\|S_{n+1}\|_{L^{p'}([0,t-1]\times R^n)} \leqslant C t^{(n-1)/2 + 1/p' - n/(2p)},$$

and using (3.24) for a lower bound on the numerator, we obtain

$$F(t) \geqslant C_0(t-1)^{p[d(p,m)-(n-1)/2-1/p'+n/(2p)]}, \qquad t \geqslant t_m, \tag{3.26}$$

where $C_0 > 0$. We estimate from below

$$\begin{aligned} d(p,m) &- (n-1)/2 - 1/p' + n/(2p) \\ &\geqslant [1/(p-1) - n/2]\, p^m - 1/(p-1) - (p-1)/p + n/(2p) \\ &> [1/(p-1) - n/2]\, p^m - (p+1)/(p-1). \end{aligned}$$

Thus, assumption (3.21) implies that the exponent in (3.26) is at least B. Therefore, we have $F(t) \geqslant C_0(t-1)^B$ for sufficiently large t. On the other hand, (3.16) gives $F(t) \geqslant \varepsilon c_0$ for all t. These estimates complete the proof of Proposition 3.2.

We conclude this section with the proofs of the auxiliary results used above.

Proof of Lemma 3.2. Let u_L be the solution of the linear problem $(\Box + \partial_t)\, u_L = 0$ with initial data $u_L|_{t=0} = u_0$, $\partial_t u_L|_{t=0} = u_1$. From the fact that $(u - \varepsilon u_L)|_{t=0} = 0$ and $\partial_t(u - \varepsilon u_L)|_{t=0} = 0$, we have the equation

$$(\Box + \partial_t)(Hu - \varepsilon H u_L) = H\, |u|^p, \qquad t \in \mathbf{R}.$$

Now, we convolve both sides with S_{2k}. Since $(\Box + \partial_t)\, S_{2k} = S_{2(k-1)}$, we obtain

$$S_{2(k-1)} * (Hu - \varepsilon H u_L) = S_{2k} * H\, |u|^p. \tag{3.27}$$

We next express $S_{2(k-1)} * Hu_L$ in terms of the initial data:

$$\begin{aligned} S_{2(k-1)} * Hu_L &= (\Box + \partial_t)\, S_{2k} * Hu_L \\ &= S_{2k} * (\Box + \partial_t)\, Hu_L \\ &= S_{2k} * (\partial_t^2 H u_L + 2\partial_t H \partial_t u_L + \partial_t H u_L + H(\Box + \partial_t)\, u_L). \end{aligned} \tag{3.28}$$

Since u_L is a solution of the linear equation and $\partial_t H = \delta_0$, we can simplify (3.28) into

$$\begin{aligned} S_{2(k-1)} * Hu_L &= \partial_t(S_{2k} * \partial_t H u_L) + S_{2k} * \partial_t H \partial_t u_L + S_{2k} * \partial_t H u_L \\ &= \partial_t S_{2k}(t) * u_0 + S_{2k}(t) * (u_0 + u_1). \end{aligned}$$

Combining this with (3.27) completes the proof.

Proof of Lemma 3.3. We begin with the asymptotics (3.19). The expression to be estimated is

$$S_{2k}(t) * [u_0 + u_1](x) = \int_{B(K)} S_{2k}(t, x-y)[u_0(y) + u_1(y)]\, dy. \quad (3.29)$$

Let us introduce $\nu = k - (n+1)/2$ and $\rho = \sqrt{t^2 - (x-y)^2}$. By (3.11), we have

$$S_{2k}(t, x-y) = C_{n,k} e^{-t/2} \rho^\nu I_\nu(\rho/2),$$

where $C_{n,k} = 1/[2^n \pi^{(n-1)/2} \Gamma(k)]$.

Moreover, for bounded $|x|$ and $|y|$, $\rho = t + \mathcal{O}(\frac{1}{t})$ as $t \to \infty$. We now use asymptotics (3.17) to obtain

$$\begin{aligned} S_{2k}(t, x-y) &= C_{n,k} e^{-t/2} \rho^\nu \frac{e^{\rho/2}}{\sqrt{\pi\rho}} \left(1 + \mathcal{O}\left(\frac{1}{\rho}\right)\right) \\ &= C_{n,k} \pi^{-1/2} t^{\nu - 1/2} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right). \end{aligned}$$

Substituting this in (3.29) gives (3.19).

Let us turn to (3.20) and consider

$$\partial_t S_{2k}(t) * u_0(x) = \int_{B(K)} \partial_t S_{2k}(t, x-y)\, u_0(y)\, dy. \quad (3.30)$$

Recall that $S_{2k}(t, x-y) = C_{n,k} e^{-t/2} \rho^\nu I_\nu(\rho/2)$. From $\partial_t \rho = t/\rho$ and the identity

$$\frac{d}{d\rho}(\rho^\nu I_\nu(\rho/2)) = \frac{1}{2} \rho^\nu I_{\nu-1}(\rho/2)$$

(cf. [13], Chapter 1), we have

$$\begin{aligned} \partial_t(e^{-t/2} \rho^\nu I_\nu(\rho/2)) &= \frac{t}{2\rho} e^{-t/2} \rho^\nu I_{\nu-1}(\rho/2) - \frac{1}{2} e^{-t/2} \rho^\nu I_\nu(\rho/2) \\ &= \frac{e^{-t/2} \rho^{\nu-1}}{2} (t I_{\nu-1}(\rho/2) - \rho I_\nu(\rho/2)). \end{aligned}$$

An application of (3.17) to the difference $tI_{\nu-1}(\rho/2)-\rho I_\nu(\rho/2)$ shows that the leading terms cancel each other, so that

$$tI_{\nu-1}(\rho/2)-\rho I_\nu(\rho/2)=\mathcal{O}\left(\frac{e^{\rho/2}}{\sqrt{\rho}}\right).$$

Therefore, $\partial_t S_{2k}(t,x)=C_{n,k}\partial_t(e^{-t/2}\rho^\nu I_\nu(\rho/2))=\mathcal{O}(t^{\nu-3/2})$. Now the asymptotics (3.20) follows from (3.30) by integrating over $B(K)$.

Proof of Lemma 3.4. We have to estimate $\|S_{2k}\|^q_{L^q([0,t]\times R^n)}$, that is

$$C^q_{n,k}\int_0^t\int_{B(s)} e^{-qs/2}(\sqrt{s^2-|x|^2})^{qk-q(n+1)/2}\, I^q_{k-(n+1)/2}(\sqrt{s^2-|x|^2}/2)\,dx\,ds,$$

where $C_{n,k}=1/[2^n\pi^{(n-1)/2}\Gamma(k)]$.

By the asymptotics (3.17) for the modified Bessel function, the above quantity is bounded by

$$C\int_0^t\int_{B(s)}(s^2-|x|^2)^{qk/2-q(n+1)/4}\,e^{-qs/2}\,\frac{e^{(q/2)(s^2-|x|^2)^{1/2}}}{1+(s^2-|x|^2)^{q/4}}\,dx\,ds.$$

Now, we use the inequality $(s^2-|x|^2)^{1/2}\leqslant s-|x|^2/(2s)$ to deduce

$$\begin{aligned}&\int_0^t e^{-qs/2}e^{qs/2}\left(\int_{B(s)}(s^2-|x|^2)^{qk/2-q(n+2)/4}\,e^{-|x|^2/(4s)}\,dx\right)ds\\&\qquad\leqslant C\int_0^t s^{qk-q(n+2)/2}\left(\int_{B(s)}e^{-|x|^2/(4s)}\,dx\right)ds.\end{aligned}$$

It is easy to see that the last integral over $B(s)$ is bounded by $(4\pi s)^{n/2}$, so we obtain the final estimate

$$\|S_{2k}\|^q_{L^q([0,t]\times R^n)}\leqslant C\int_0^t s^{qk-q(n+2)/2+n/2}\,ds\leqslant Ct^{q(k-1)-(q-1)\,n/2+1}.$$

The proof of Lemma 3.4 is complete.

APPENDIX

We will derive properties of S_{2k} from those of the *Riesz distribution* ${}_mZ_{2k}$ associated to the Klein–Gordon operator $\Box+m^2$ (cf. [10], Chapter 10).

Let $C_+ = \{(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n : |x| < t\}$ be the forward light cone. By definition,

$$
{}_mZ_{2k}(t, x) = \begin{cases} \dfrac{(m^{-1}\sqrt{t^2 - |x|^2})^{k-(n+1)/2} J_{k-(n+1)/2}(m\sqrt{t^2 - |x|^2})}{\pi^{(n-1)/2} 2^{k+(n-1)/2} \Gamma(k)}, \\ \qquad (t, x) \in C_+, \\ 0, \qquad (t, x) \notin C_+, \end{cases} \tag{A.31}
$$

where Γ and $J_{k-(n+1)/2}$ are the Gamma function and Bessel function of order $k-(n+1)/2$ respectively. It is known that ${}_mZ_{2k}$ is an entire complex function of the pair (m, k) valued in $\mathscr{D}'(\bar{C}_+)$. Among its properties, the most important are

$$
\begin{aligned} {}_mZ_{2k} * {}_mZ_{2l} &= {}_mZ_{2(k+l)}, \qquad \forall k, l \in \mathbf{C}, \\ (\Box + m^2)\, {}_mZ_{2k} &= {}_mZ_{2(k-1)}, \qquad \text{and} \qquad {}_mZ_0 = \delta_0. \end{aligned} \tag{A.32}
$$

Noting that $\Box + \partial_t = e^{-t/2}(\Box - \frac{1}{4})\, e^{t/2}$, we may use the distribution ${}_mZ_{2k}$ to construct S_{2k} by setting $S_{2k} = e^{-t/2}\, {}_{i/2}Z_{2k}$. In fact, a straightforward calculation shows that the counterparts of (A.32), namely (3.12), hold.

It is convenient to express S_{2k} in terms of the modified Bessel function $I_{k-(n+1)/2}$: since $I_\nu(z) = e^{-i\nu\pi/2} J_\nu(iz)$ for each ν, z, we obtain (3.11) from (A.31). From this representation, we easily see that S_{2k} is positive and continuous for $k \geqslant (n+1)/2$.

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