

Critical Exponent for Semilinear Wave Equations with Space-Dependent Potential

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December 3, 2007

Abstract

We study the balance between the effect of spatial inhomogeneity of the potential in the dissipative term and the focusing nonlinearity. Sharp critical exponent results will be presented in the case of slow decaying potential.

1 Introduction

We consider the following Cauchy problem for the semilinear damped wave equation

$$u_{tt}(t, x) - \Delta u(t, x) + V(x)u_t(t, x) = |u(t, x)|^p, \quad (t, x) \in (0, \infty) \times \mathbf{R}^N, \quad (1.1)$$

$$u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x), \quad x \in \mathbf{R}^N, \quad (1.2)$$

where $\varepsilon > 0$, (u_0, u_1) are compactly supported initial data from the energy space

$$u_0 \in H^1(\mathbf{R}^N), \quad u_1 \in L^2(\mathbf{R}^N),$$

$V \in L^\infty(\mathbf{R}^N)$ is a potential function specified later, and the power p of the nonlinearity satisfies

$$1 < p < \frac{N+2}{N-2} \quad (N \geq 3), \quad 1 < p < +\infty \quad (N = 1, 2).$$

Such equations appear in models for travelling waves in a nonhomogeneous gas with damping that changes with the position. $V(x)$ is referred as a friction coefficient or potential (see Ikawa [7]).

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Keywords and Phrases: Damped wave equation; Subcritical potential; Energy decay; L^2 -decay; Small data; Global existence and blow-up; Finite propagation speed.

2000 Mathematics Subject Classification. Primary 35L70; Secondary 35L05, 35B33, 35B40.

Our interest is focused on the so-called critical exponent $p_c(N)$, which is a number defined by the following property:

If $p_c(N) < p$, **all small data solutions of (1.1)-(1.2) are global**;

while if $1 < p \leq p_c(N)$ all solutions of (1.1)-(1.2) with data positive on average **blow-up in finite time regardless of the smallness of the data**.

Our goal in this paper is to solve the critical exponent problem for the equation (1.1).

It is well known that if the damping is missing, $V(x) = 0$, the critical exponent $p_w(N)$ for the wave equation $(\partial_{tt} - \Delta)u = |u|^p$ is the positive root of $(N-1)p^2 - (N+1)p - 2 = 0$, where $N \geq 2$ is the space dimension ($p_w(1) = \infty$, see Sideris [27]). The proof of this fact, famous as Strauss' conjecture [29], took almost 20 years and the effort of many mathematicians, beginning with John [14], Glassey [3], Sideris [28], Strauss [30], Zhou [40], and ending with Lindblad and Sogge [18], Georgiev, Lindblad and Sogge [2] and Tataru [31]. The number $p_w(N)$ is referred as **Strauss critical exponent**.

In [32] and [33], Todorova and Yordanov solved the critical exponent problem for the wave equation (1.1) when the potential $V(x)$ is a constant. The main result is that the critical exponent $p_c(N)$ of the damped wave equation (1.1) with $V(x) = \text{const}$ is exactly $1 + 2/N$. The number $p_c(N) = 1 + 2/N$ is the famous **Fujita's critical exponent for the heat equation** $\partial_t v - \Delta v = v^p$ (see [1]). More precisely, they prove small data global existence for the damped wave equation (1.1) with $V(x) = \text{const}$ and exponent $p > 1 + 2/N$. If $1 < p < 1 + 2/N$, they prove blow up for all solutions of (1.1) with data positive on average. In [33] the authors prove other results which also indicate parabolic asymptotic profile for solutions of equation (1.1) with constant potential $V(x)$ for large exponent $p > p_c(N)$. Later on Zhang [39] proved that the critical exponent $1 + 2/N$ belongs to the blow up region. Ikehata-Tanizawa [13] consider the global existence part for noncompactly supported data. There are many related results to the so-called diffusion phenomenon, and we quote some of them: Han-Milani [4], Ikehata [8], Ikehata-Miyaoka-Nakatake [9], Ikehata-Nishihara [10], Ikehata-Ohta [12], Hayashi-Kaikina-Naumkin [5], Hosono-Ogawa [6], Marcati-Nishihara [19], Nishihara [24, 25], Li-Zhou [17], Narazaki [23], Zhang [39], and the references there in.

In this paper we solve the critical exponent problem for the equation (1.1)-(1.2) under natural conditions for the potential $V(x)$. For the sake of simplicity the conditions for the potential $V(x) > 0$ are the following:

(A) $V(x) \in C^1(\mathbb{R}^N)$ is a radially symmetric function

$$V(x) \sim V_0(1 + |x|)^{-\alpha}, \quad |x| \rightarrow \infty,$$

with $V_0 > 0$ and $\alpha \in [0, 1)$.

In the **case of slow decaying potential** $0 \leq \alpha < 1$ **in (A)** we prove that the critical exponent for the problem (1.1)-(1.2) is

$$p_c(N, \alpha) = 1 + \frac{2}{N - \alpha}.$$

The case of fast decaying potential, namely the exponent $\alpha > 1$ in (A). In this case there is no decay of the energy of the linear part of equation (1.1). The result of Mochizuki [21] says that the energy of the linear part of equation (1.1) approaches a non-zero constant as $t \rightarrow \infty$ if $V(x) = O(|x|^{-1-\delta})$ with $\delta > 0$. In this case we expect that equation (1.1) loses its "parabolicity" asymptotic effects and turns back to the regime of pure wave equation. Respectively, we expect that the critical exponent $p_c(N, \alpha)$ of the damped wave equation in

the case of fast decaying potential $\alpha > 1$ jumps to the critical exponent of the wave equation–Strauss’ number $p_w(N)$. Namely, $p_c(N, \alpha) = p_w(N)$ for any $\alpha > 1$. The proof of both parts – small data global existence and blow-up part is quite involved and will be presented elsewhere.

The case of critically decaying potential $\alpha = 1$ in **(A)** is very delicate. This case is a transition from asymptotic parabolicity to the pure hyperbolic regime. The energy decay of the linear part of equation (1.1) depends in a very interesting way on the coefficient V_0 of the potential $V(x)$. Correspondingly, the critical exponent is $1 + \frac{1}{N-1}$ for large $V_0 \geq N - 1$, while for $0 < V_0 < N - 1$ the critical exponent $p_c(N, V_0)$ increases when $V_0 \rightarrow 0$. These results will be presented elsewhere.

In this paper we show further interesting phenomena due to the presence of the damping term. We also derive the exact decay rate for the energy and L^2 and L^{p+1} norms of solutions in the global existence part for exponents of nonlinearity $p > p_c(N, \alpha)$.

To get a sharp critical exponent result we need sharp decay estimates for the linear problem

$$u_{tt} - \Delta u + V(x)u_t = 0. \quad (1.3)$$

This problem is quite delicate in the case of space dependent potential. Recently Todorova-Yordanov [34, 35] derived an almost optimal decay for solutions of (1.3). The key idea is that we are able to derive asymptotically a very good approximation for the fundamental solution of the equation (1.3). In this paper, for the global existence part namely, the case $p_c(N, \alpha) < p$ we use a modification of the approach in [34], [35]. For the blow up the part of (1.1)-(1.2) namely $1 < p \leq p_c(N, \alpha)$ we apply the method of the test functions developed by Zhang [37, 38, 39].

Now we are ready to state our main results.

Denote by $X_1(0, T) := C([0, T]; H^1(\mathbf{R}^N)) \cap C^1([0, T]; L^2(\mathbf{R}^N))$.

Our small data global existence results read as follows.

Theorem 1.1 *Let $V(x)$ satisfies the condition **(A)**; $\alpha \in [0, 1)$ and $p_c(N, \alpha) < p < \frac{N+2}{N-2}$ for $N \geq 3$; and $p_c(N, \alpha) < p < \infty$ for $N = 1, 2$. Then there exists a number $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the problem (1.1)-(1.2) has a unique solution $u \in X_1(0, +\infty)$ satisfying*

$$\int_{\mathbf{R}^N} e^{\frac{(\kappa-\delta)|x|^2-\alpha}{t}} u(t, x)^2 dx \leq Ct^{-m+\frac{\alpha}{2-\alpha}},$$

$$\int_{\mathbf{R}^N} e^{\frac{\kappa|x|^2-\alpha}{t}} (u_t(t, x)^2 + |\nabla u(t, x)|^2) dx \leq Ct^{-m-1},$$

$$\int_{\mathbf{R}^N} e^{\frac{\kappa|x|^2-\alpha}{t}} |u(t, x)|^{p+1} dx \leq Ct^{-m-1-\gamma},$$

for large $t \gg 1$, where

$$\kappa = \frac{(m-\delta)V_1}{(2-\alpha)(N-\alpha)},$$

$$m = \frac{N-\alpha}{2-\alpha} - 2\delta,$$

$$\gamma = \frac{(p-1)(N-\alpha)-2}{2-\alpha} - (p-1)\delta > 0$$

and $\delta > 0$ is a small number.

Corollary 1.1 *Under the same assumptions as in Theorem 1.1 the following estimates hold*

$$\int_{\mathbf{R}^N} u(t, x)^2 dx \leq Ct^{-(\frac{N-\alpha}{2-\alpha} - \frac{\alpha}{2-\alpha} - 2\delta)},$$

$$\int_{\mathbf{R}^N} (u_t(t, x)^2 + |\nabla u(t, x)|^2) dx \leq Ct^{-(\frac{N-\alpha}{2-\alpha} + 1 - 2\delta)},$$

$$\int_{\mathbf{R}^N} |u(t, x)|^{p+1} dx \leq Ct^{-(p\frac{N-\alpha}{2-\alpha} - \frac{\alpha}{2-\alpha} - (p+1)\delta)},$$

for large $t \gg 1$, and small $\delta > 0$.

Another important consequence of the main theorem is the following.

Corollary 1.2 *Under the same assumptions as in Theorem 1.1, the solution of (1.1)-(1.2) satisfies*

$$\int_{|x|^{2-\alpha} \geq t^{1+\rho}} (u_t(t, x)^2 + |\nabla u(t, x)|^2) dx \leq C_R t^{-m} e^{-\kappa t^\rho}$$

for an arbitrarily fixed $\rho > 0$.

Namely, the decay rate of the energy under consideration in the region $|x|^{2-\alpha} \geq t^{1+\rho}$ ($\rho > 0$), is exponential. This shows parabolic asymptotic profile of solutions of the problem (1.1)-(1.2).

The blowup result in the case when $1 < p \leq p_c(N, \alpha)$ is as follows.

Theorem 1.2 *Let the potential $V(x)$ satisfies the condition **(A)**; $\alpha \in [0, 1)$ and $1 < p \leq p_c(N, \alpha)$. If the initial data (u_0, u_1) satisfy*

$$\int_{\mathbf{R}^N} (u_1(x) + V(x)u_0(x)) dx > 0,$$

then the solution of problem (1.1)-(1.2) does not exist globally for any $\varepsilon > 0$.

This paper is organized as follows. In section 2 we shall prove Theorem 1.1 by dividing the proof into several lemmas, and section 3 is devoted to the proof of Theorem 1.2.

2 Global existence of small amplitude solutions

The following classical local existence result for the problem (1.1)-(1.2) is a simple modification of the result in Strauss [30].

Then, under the assumption **(A)** for the potential $V(x)$, for any compactly supported data (u_0, u_1) from the energy space

$$u_0 \in H^1(\mathbf{R}^N), \quad u_1 \in L^2(\mathbf{R}^N), \quad \text{supp } u_i \subset B(R) := \{x \in \mathbf{R}^N : |x| < R\}, \quad i = 0, 1,$$

the problem (1.1)-(1.2) admits a unique local solution $u \in X_1(0, T_m)$ for some $T_m = T_m(\varepsilon) \in (0, +\infty]$, and if $T_m < +\infty$, then

$$\limsup_{t \uparrow T_m} (\|u_t(t, \cdot)\| + \|\nabla u(t, \cdot)\|) = +\infty. \quad (2.1)$$

Moreover, the finite propagation speed property holds:

$$\text{supp } u(t, \cdot) \subset B(R + t), \quad t \in [0, T_m).$$

A starting point of the proof of global existence of small data solutions fully depends on the previous work due to [34]. Indeed, for solution $u(t, x)$ on $[0, T_m(\varepsilon))$ of the problem (1.1)-(1.2) we set $v = uw^{-1}$, where w is an approximate solution of the linear part of (1.1)-(1.2) and can be defined by

$$w(t, x) := t^{-m} e^{-m_1 \phi(x)/t}. \quad (2.2)$$

The parameters m, m_1 are determined below. Here $\phi \in C^2(\mathbf{R}^N)$ is a positive solution for the Poisson equation:

$$\Delta \phi(x) = V(x), \quad x \in \mathbf{R}^N,$$

satisfying

$$\begin{aligned} \phi_0(1 + |x|)^{2-\alpha} &\leq \phi(x) \leq \phi_1(1 + |x|)^{2-\alpha}, \\ m(V) &:= \liminf_{|x| \rightarrow +\infty} \frac{V(x)\phi(x)}{|\nabla \phi(x)|^2} > 0 \end{aligned}$$

where $\phi_i > 0$ ($i = 0, 1$) are constants depending on V_0, N and α . Solutions $\phi(x)$ with the above properties exist in many cases, including the radial potential $V(x)$ which satisfies the condition **(A)**. In the later case $m(V)$ can be calculated explicitly as follows

$$m(V) = \frac{N - \alpha}{2 - \alpha},$$

(see [35, Proposition 1.1, Example 1.2]). In this case

$$m := m(V) - 2\delta; \quad m_1 := m(V) - \delta,$$

where $\delta > 0$ is a small number.

We also set

$$P(t, x) := \frac{3}{4} \left(\frac{6}{t} + \frac{S(x)}{t^2} \right)^{-1} w(t, x),$$

where

$$S(x) := (m(V) - \delta)\phi(x).$$

Note that such P and w are defined independently of the solution itself, and satisfy the following properties.

Lemma 2.1 *Let $\alpha \in [0, 1)$ as in condition **(A)**. There exists a large number $t_0 > 0$ such that for $t \geq t_0$ the following inequalities hold*

- (i) $C_1 P(t, x) \geq t^\alpha w(t, x)$,
 - (ii) $|w_t(t, x)| \leq C_1 t^{-\alpha} w(t, x)$,
- with some constant $C_1 > 0$.

Proof. We first obtain

$$\begin{aligned} P t^{-\alpha} w^{-1} &= \frac{3}{4} \frac{1}{6t + (m(V) - \delta)\phi(x)} t^{2-\alpha} \\ &\geq \frac{3}{4} t^{2-\alpha} \frac{1}{6t + C_0(1 + |x|)^{2-\alpha}} \geq \frac{3}{4} \frac{t^{2-\alpha}}{6t + C_0(1 + t + R)^{2-\alpha}} \end{aligned}$$

with some constant $C_0 > 0$. Since

$$\lim_{t \rightarrow +\infty} \frac{t^{2-\alpha}}{6t + C_0(1 + t + R)^{2-\alpha}} = \frac{1}{C_1} > 0,$$

one has the property **(i)**. **(ii)** is also similar.

From the condition **(A)** on the potential $V(x)$ it elementary follows that there exist constants $V_2, V_1 > 0$ such that the following point wise estimate holds

$$V_1(1 + |x|)^{-\alpha} \leq V(x) \leq V_2(1 + |x|)^{-\alpha}. \quad (2.3)$$

It is easy to check that $v = uw^{-1}$ satisfies the transformed equation on $[0, T_m(\varepsilon))$:

$$v_{tt} - \Delta v + (V + 2w^{-1}w_t)v_t + (-2w^{-1}\nabla w) \cdot \nabla v + Q(t, x)v = w^{p-1}|v|^p, \quad (2.4)$$

where

$$Q(t, x) = w^{-1}(w_{tt} - \Delta w + Vw_t).$$

Multiplying both sides of (2.4) by $(Pv_t + wv)$, and integrating over \mathbf{R}^N we have

$$\frac{d}{dt}E(v_t, \nabla v)(t) + F(t) + G(t) = H(t) + \frac{d}{dt}\left\{\frac{1}{p+1} \int_{\mathbf{R}^N} Pw^{p-1}|v|^{p+1}dx\right\}, \quad t \in [0, T_m(\varepsilon)), \quad (2.5)$$

where the weighted energy $E(v_t, \nabla v)(t)$ is

$$E(v_t, \nabla v)(t) = \frac{1}{2} \int_{\mathbf{R}^N} \{P(v_t^2 + |\nabla v|^2) + 2wv_t v + (QP + w_t + Vw)v^2\}dx, \quad (2.6)$$

and

$$\begin{aligned} F(t) &= \frac{1}{2} \int_{\mathbf{R}^N} (-P_t + 2VP + 4P(\log w)_t - 2w)v_t^2 dx \\ &+ \int_{\mathbf{R}^N} (\nabla P - 2P\nabla \log w) \cdot v_t \nabla v dx + \frac{1}{2} \int_{\mathbf{R}^N} (-P_t + 2w)|\nabla v|^2 dx, \\ G(t) &= \frac{1}{2} \int_{\mathbf{R}^N} [Qw - (QP)_t]v^2 dx, \\ H(t) &= \int_{\mathbf{R}^N} \left\{w^p - \frac{1}{p+1}(Pw^{p-1})_t\right\}|v|^{p+1} dx. \end{aligned} \quad (2.7)$$

Lemma 2.2 *Let $\alpha \in [0, 1)$ be the exponent of the potential in condition **(A)**. There exists a large number $t_0 > 0$ such that for $t \geq t_0$, and for small $\delta > 0$ the following estimates hold*

(iii) $-P_t + w \geq 0,$

(iv) $Q \geq 0, Q_t \leq 0, Qw - (QP)_t = Q(w - P_t) - Q_t P \geq 0,$

(v) $(-P_t + 2w)(-P_t + 2VP + 4P(\log w)_t - 2w) \geq |\nabla P - 2P\nabla \log w|^2.$

Proof. The proof is similar to [34, 35], and so we omit it.

Remark 2.1 By taking $\varepsilon > 0$ sufficiently small we can make the life span $T_m = T_m(\varepsilon)$ of the solution of (1.1)-(1.2) large enough such that $T_m(\varepsilon) > t_0$, where $t_0 > 0$ is the time defined in Lemmas 2.1 and 2.2.

By using Lemma 2.2 we see that $F(t)$ in (2.7) is a positive definite quadratic form, therefore $F(t) \geq 0$. Also $G(t) \geq 0$ because of condition **(iv)**. Therefore, after integrating (2.5) over $[t_0, t]$, where $t_0 < t < T_m$, we have

$$E(v_t, \nabla v)(t) \leq E(v_t, \nabla v)(t_0) + \frac{1}{p+1} \int_{\mathbf{R}^N} Pw^{p-1}|v|^{p+1}dx + \int_{t_0}^t H(s)ds. \quad (2.8)$$

We need estimates for the second and third terms of the right hand side of the estimate (2.8). Now we introduce a new function:

$$W(t) := \int_{\mathbf{R}^N} P(t, x)(v_t^2 + |\nabla v|^2)dx + \int_{\mathbf{R}^N} V(x)w(t, x)v(t, x)^2 dx.$$

For the second term of the right hand side of (2.8) we have the following crucial estimate.

Lemma 2.3 *Let $V(x)$ satisfies the condition **(A)** and $\alpha \in [0, 1)$. If $p_c(N, \alpha) < p$, then there is a number $\gamma > 0$, which depends on p, N, α and δ such that*

$$\int_{\mathbf{R}^N} \left(1 + \frac{S(x)}{t}\right) P w^{p-1} |v|^{p+1} dx \leq C t^{-\gamma} W(t)^{(p+1)/2}, \quad t \in [t_0, T_m).$$

Proof. From definitions of $w(t, x)$ and $P(t, x)$ we have

$$P w^{p-1} |v|^{p+1} = \frac{3}{4} \left(\frac{6}{t} + \frac{S}{t^2}\right)^{-1} w^p |v|^{p+1} \leq C t w^p |v|^{p+1},$$

and

$$\frac{S(x)}{t} P w^{p-1} |v|^{p+1} = \frac{3}{4} t \frac{S(x)}{t} \left(6 + \frac{S(x)}{t}\right)^{-1} w^p |v|^{p+1} \leq \frac{3}{4} t w^p |v|^{p+1}.$$

Therefore,

$$\int_{\mathbf{R}^N} \left(1 + \frac{S(x)}{t}\right) P w^{p-1} |v|^{p+1} dx \leq C t \int_{\mathbf{R}^N} w^p |v|^{p+1} dx = C t^{-(pm-1)} \int_{\mathbf{R}^N} e^{-(pS(x))/t} |v|^{p+1} dx. \quad (2.9)$$

By setting

$$\psi(t, x) = \frac{1}{2} \frac{S(x)}{t}, \quad \eta = \frac{2p}{p+1},$$

we can rewrite (2.9) in the form

$$\int_{\mathbf{R}^N} \left(1 + \frac{S(x)}{t}\right) P w^{p-1} |v|^{p+1} dx \leq C t^{-(pm-1)} \|e^{-\eta\psi(t, \cdot)} v\|_{p+1}^{p+1}. \quad (2.10)$$

To estimate the weighted norm $\|e^{-\eta\psi(t, \cdot)} v\|_{p+1}$ we use Gagliardo-Nirenberg inequality and get

$$\|e^{-\eta\psi(t, \cdot)} v\|_{p+1} \leq \|e^{-\eta\psi(t, \cdot)} v\|^\theta \|\nabla(e^{-\eta\psi(t, \cdot)} v)\|^{1-\theta}, \quad (2.11)$$

where

$$\theta = 1 - N \left(\frac{1}{2} - \frac{1}{p+1}\right).$$

Since

$$|\nabla(e^{-\eta\psi} v)|^2 = \eta^2 e^{-2\eta\psi} |\nabla\psi|^2 v^2 - 2\eta e^{-2\eta\psi} v \nabla\psi \cdot \nabla v + e^{-2\eta\psi} |\nabla v|^2,$$

integrating by parts we obtain

$$\begin{aligned} 2\eta \int_{\mathbf{R}^N} e^{-2\eta\psi} v \nabla v \cdot \nabla\psi dx &= \eta \int_{\mathbf{R}^N} e^{-2\eta\psi} \nabla v^2 \cdot \nabla\psi dx = \eta \int_{\mathbf{R}^N} (e^{-2\eta\psi} \nabla\psi) \cdot \nabla v^2 dx \\ &= -\eta \int_{\mathbf{R}^N} v^2 (e^{-2\eta\psi} \Delta\psi - 2\eta e^{-2\eta\psi} |\nabla\psi|^2) dx \\ &= 2\eta^2 \int_{\mathbf{R}^N} e^{-2\eta\psi} v^2 |\nabla\psi|^2 dx - \eta \int_{\mathbf{R}^N} v^2 e^{-2\eta\psi} \Delta\psi dx. \end{aligned}$$

Thus we find

$$\begin{aligned} \int_{\mathbf{R}^N} |\nabla(e^{-\eta\psi} v)|^2 dx &= \int_{\mathbf{R}^N} e^{-2\eta\psi} |\nabla v|^2 dx + \int_{\mathbf{R}^N} e^{-2\eta\psi} (\eta \Delta\psi - \eta^2 |\nabla\psi|^2) v^2 dx \\ &\leq \int_{\mathbf{R}^N} e^{-2\eta\psi} |\nabla v|^2 dx + \eta \int_{\mathbf{R}^N} e^{-2\eta\psi} (\Delta\psi) v^2 dx. \end{aligned}$$

Therefore, we can rewrite (2.11), as follows

$$\|e^{-\eta\psi(t)}v\|_{p+1} \leq \|e^{-\eta\psi(t)}v\|^\theta \left(\int_{\mathbf{R}^N} e^{-2\eta\psi} |\nabla v|^2 dx + \eta \int_{\mathbf{R}^N} e^{-2\eta\psi} (\Delta\psi)v^2 dx \right)^{(1-\theta)/2}. \quad (2.12)$$

Now we estimate the right hand side of (2.12). From the definition of $\psi(t, x)$ and $\phi(x)$ we get

$$\Delta\psi(t, x) = \frac{1}{2t} \Delta S(x) = \frac{m(V) - \delta}{2t} \Delta\phi(x) = \frac{m(V) - \delta}{2t} V(x).$$

Thus we have

$$\int_{\mathbf{R}^N} e^{-2\eta\psi} (\Delta\psi)v(t, x)^2 dx = \frac{m(V) - \delta}{2t} \int_{\mathbf{R}^N} V(x) e^{-2\eta\psi(t, x)} v(t, x)^2 dx.$$

Since

$$e^{-2\eta\psi(t, x)} = e^{-\frac{2p}{p+1} \frac{S(x)}{t}} = e^{-\frac{S(x)}{t}} t^{-m} e^{-\frac{(p-1)S(x)}{(p+1)t}} t^m \leq t^m w(t, x), \quad (2.13)$$

we get

$$\int_{\mathbf{R}^N} e^{-2\eta\psi} (\Delta\psi)v(t, x)^2 dx \leq C t^{m-1} \int_{\mathbf{R}^N} V(x) w(t, x) v(t, x)^2 dx. \quad (2.14)$$

To estimate the weighted norm $\|e^{-\eta\psi(t)}v\|$ in (2.12) we use the following decomposition:

$$\begin{aligned} e^{-2\eta\psi(t, x)} v(t, x)^2 &= e^{-S(x)/t} t^{-m} t^m e^{-(S(x)(p-1))/((p+1)t)} t^{-\alpha/(2-\alpha)} t^{\alpha/(2-\alpha)} v^2 \\ &= t^m w(t, x) v(t, x)^2 (e^{-(S(x)(p-1))/((p+1)t)} t^{-\alpha/(2-\alpha)}) t^{\alpha/(2-\alpha)}. \end{aligned} \quad (2.15)$$

Furthermore, there exists a constant $C > 0$ such that for any $x > 0$ it is true that

$$x^{\alpha/(2-\alpha)} \leq C e^{x(p-1)/(p+1)}, \quad \alpha \in [0, 1), \quad \frac{\alpha}{2-\alpha} < 1.$$

So, with a generous constant $C > 0$ we have

$$\begin{aligned} C e^{-(p-1)S(x)/(p+1)t} t^{-\alpha/(2-\alpha)} &\leq C t^{-\alpha/(2-\alpha)} \left(\frac{S(x)}{t}\right)^{-\alpha/(2-\alpha)} \\ &= C S(x)^{-\alpha/(2-\alpha)} = C (m(V) - \delta)^{-\alpha/(2-\alpha)} \phi(x)^{-\alpha/(2-\alpha)} \leq C V(x). \end{aligned}$$

This inequality combined with (2.15) implies

$$e^{-2\eta\psi(t, x)} v(t, x)^2 \leq C t^{\alpha/(2-\alpha)+m} w(t, x) v(t, x)^2 V(x),$$

which shows the desired estimate

$$\|e^{-\eta\psi(t)}v\|^2 \leq C t^{m+(\alpha/(2-\alpha))} \int_{\mathbf{R}^N} V(x) w(t, x) v(t, x)^2 dx. \quad (2.16)$$

On the other hand, in order to estimate the quantity $\|e^{-\eta\psi(t)}\nabla v\|$ of (2.12) we use (2.13). In fact, from the definition of P and (2.13) we see that

$$\begin{aligned} e^{-2\eta\psi(t, x)} &= t^m w(t, x) e^{-\frac{p-1}{p+1} \frac{S(x)}{t}} = t^m \frac{4}{3} \left(\frac{6}{t} + \frac{S(x)}{t^2} \right) P(t, x) e^{-\frac{p-1}{p+1} \frac{S(x)}{t}} \\ &= \frac{4}{3} t^{m-1} P(t, x) \left\{ \left(6 + \frac{S(x)}{t} \right) e^{-\frac{p-1}{p+1} \frac{S(x)}{t}} \right\} \leq C t^{m-1} P(t, x), \end{aligned}$$

where $C > 0$ is a constant determined by the fact that the function $x \mapsto (6+x)e^{-Kx}$ ($K > 0$) is bounded above, that is, for all $x \geq 0$, $(6+x)e^{-Kx} \leq C$. This implies

$$\int_{\mathbf{R}^N} e^{-2\eta\psi} |\nabla v|^2 dx \leq Ct^{m-1} \int_{\mathbf{R}^N} P(t, x) |\nabla v|^2 dx. \quad (2.17)$$

Therefore, by using (2.12), (2.14), (2.16) and (2.17) we have

$$\begin{aligned} \|e^{-\eta\psi(t)} v\|_{\frac{p+1}{p+1}}^{p+1} &\leq C \left(t^{m+(\alpha/(2-\alpha))} \int_{\mathbf{R}^N} V(x) w(t, x) v(t, x)^2 dx \right)^{(p+1)\theta/2} \\ &\times \left(t^{(m-1)} \int_{\mathbf{R}^N} V(x) w(t, x) v(t, x)^2 dx + t^{(m-1)} \int_{\mathbf{R}^N} P(t, x) |\nabla v(t, x)|^2 dx \right)^{(p+1)(1-\theta)/2} \\ &\leq C \left(t^{m+(\alpha/(2-\alpha))} W(t) \right)^{(p+1)\theta/2} \left(t^{(m-1)} W(t) \right)^{(p+1)(1-\theta)/2} \\ &= Ct^{\{(m+\frac{\alpha}{2-\alpha})\frac{\theta}{2} + (m-1)\frac{1-\theta}{2}\}(p+1)} W(t)^{\frac{p+1}{2}} \quad (t > t_0). \end{aligned}$$

This inequality together with (2.10) gives

$$\int_{\mathbf{R}^N} \left(1 + \frac{S(x)}{t}\right) P w^{p-1} |v|^{p+1} dx \leq Ct^{-\gamma} W(t)^{\frac{p+1}{2}}, \quad t > t_0,$$

where

$$\begin{aligned} \gamma &:= (pm - 1) - \left\{ \left(m + \frac{\alpha}{2-\alpha}\right) \frac{\theta}{2} + (m-1) \frac{1-\theta}{2} \right\} (p+1) \\ &= \frac{(p-1)(N-\alpha) - 2}{2-\alpha} - (p-1)\delta. \end{aligned}$$

Finally, from the assumption $p_c(N, \alpha) < p$ we find $\gamma > 0$, which implies the desired estimate.

Now by using the result in Lemma 2.3 we are able to estimate the third term of the right hand side of (2.8) as follows

Lemma 2.4 *Under the assumptions in Lemma 2.3 we have*

$$\int_{t_0}^t H(s) ds \leq C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m)$$

with some constant $C > 0$ and $\gamma > 0$ is the constant determined in Lemma 2.3.

Proof. It follows from the definitions of P and w that

$$K(t, x) := w^p - \frac{1}{p+1} (Pw^{p-1})_t = \frac{1}{p+1} Pw^{p-1} \left\{ (p+1) \frac{w}{P} - \frac{P_t}{P} - (p-1) \frac{w_t}{w} \right\}.$$

Since

$$\begin{aligned} \frac{P_t}{P} &= \left(\frac{6}{t} + \frac{S(x)}{t^2} \right)^{-1} \left(\frac{6}{t^2} + \frac{2S(x)}{t^3} \right) + \frac{w_t}{w}, \\ \frac{w_t}{w} &= -\frac{m}{t} + \frac{S(x)}{t^2}, \quad \frac{w}{P} = \frac{4}{3} \left(\frac{6}{t} + \frac{S(x)}{t^2} \right), \end{aligned}$$

one has

$$K(t, x) = Pw^{p-1} \left\{ \frac{4}{3} \left(\frac{6}{t} + \frac{S(x)}{t^2} \right) - \frac{p}{p+1} \left(\frac{S(x)}{t^2} - \frac{m}{t} \right) - \frac{1}{p+1} \left(\frac{6}{t} + \frac{S(x)}{t^2} \right)^{-1} \left(\frac{6}{t^2} + \frac{2S(x)}{t^3} \right) \right\}$$

$$\begin{aligned} &\leq Pw^{p-1}\left\{\frac{4}{3}\left(\frac{6}{t} + \frac{S(x)}{t^2}\right) + \frac{pm}{(p+1)t}\right\} = t^{-1}Pw^{p-1}\left\{\left(8 + \frac{pm}{p+1}\right) + \frac{4}{3}\frac{S(x)}{t}\right\} \\ &\leq Ct^{-1}Pw^{p-1}\left(1 + \frac{S(x)}{t}\right), \end{aligned}$$

which implies

$$w(t, x)^p - \frac{1}{p+1} \left(P(t, x)w(t, x)^{p-1}\right)_t \leq Ct^{-1}\left(1 + \frac{S(x)}{t}\right)P(t, x)w(t, x)^{p-1}.$$

This shows

$$\begin{aligned} \int_{t_0}^t H(s)ds &\leq \int_{t_0}^t \int_{\mathbf{R}^N} \left(w(s, x)^p - \frac{1}{p+1} (P(s, x)w(s, x)^{p-1})_t \right) |v(s, x)|^{p+1} dx ds \quad (2.18) \\ &\leq C \int_{t_0}^t s^{-1} \int_{\mathbf{R}^N} \left(1 + \frac{S(x)}{s}\right) P(s, x)w(s, x)^{p-1} |v(s, x)|^{p+1} dx ds. \end{aligned}$$

Thus, by using Lemma 2.3 and the estimate (2.18) we derive the estimate in Lemma 2.4.

From Lemmas 2.3, 2.4 and the weighted energy inequality (2.8) we get the following estimate

$$\begin{aligned} E(v_t, \nabla v) &= \frac{1}{2} \int_{\mathbf{R}^N} P(t, x)(v_t^2 + |\nabla v|^2) dx + \frac{1}{2} \int_{\mathbf{R}^N} [2wv_t v + (QP + w_t + Vw)v^2] dx \\ &\leq E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \geq t_0. \quad (2.19) \end{aligned}$$

Since $w > 0$ satisfies

$$w_{tt} - \Delta w + Vw_t \geq 0,$$

(see [35, Proposition 5.3]) and $P > 0$, then $QP \geq 0$. Since $\frac{d}{dt}(wv^2) - w_tv^2 = 2wv_t v$, we can rewrite (2.19) as follows

$$\begin{aligned} &\frac{1}{2} \int_{\mathbf{R}^N} P(t, x)(v_t^2 + |\nabla v|^2) dx + \frac{1}{2} \int_{\mathbf{R}^N} Vwv^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^N} (wv^2) dx \\ &\leq E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \geq t_0. \quad (2.20) \end{aligned}$$

We need one more preparation.

Lemma 2.5 *Let $\alpha \in [0, 1)$, $c_0 > 0$, $a > 0$, $R > 0$ and $E_0 > 0$ be given real numbers, and let $f \in C([t_0, T_m])$ be a monotone increasing function. If a function $h \in C^1([t_0, T_m])$ satisfies the inequality*

$$h'(t) + \frac{c_0}{(a+t+R)^\alpha} h(t) \leq E_0 + f(t),$$

then the following estimate holds

$$h(t) \leq h(t_0) + C(E_0 + f(t))(a+t+R)^\alpha, \quad t \geq t_0,$$

with some constant $C > 0$.

Proof. Since

$$\frac{d}{dt}\{e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t)\} = e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}(h'(t) + \frac{c_0}{(a+t+R)^\alpha}h(t)),$$

one obtains

$$\frac{d}{dt}\{e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t)\} \leq e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}(E_0 + f(t)).$$

By integrating over $[t_0, t]$ ($t < T_m$) one has

$$\begin{aligned} e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t) &\leq e^{\frac{c_0}{1-\alpha}(a+t_0+R)^{1-\alpha}}h(t_0) + E_0 \int_{t_0}^t e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}ds + \int_{t_0}^t e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}f(s)ds \\ &\leq e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t_0) + (E_0 + f(t)) \int_{t_0}^t e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}ds, \end{aligned}$$

where one has just used the monotone increasingness of the function $f(t)$. Since

$$\begin{aligned} \int_{t_0}^t e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}ds &= \int_{a+t_0+R}^{a+t+R} e^{C_1\tau^{1-\alpha}}d\tau \\ &= \frac{1}{1-\alpha} \int_{(a+t_0+R)^{1-\alpha}}^{(a+t+R)^{1-\alpha}} e^{C_1z}z^{\frac{\alpha}{1-\alpha}}dz \leq e^{C_1(a+t+R)^{1-\alpha}} \frac{(a+t+R)^\alpha}{C_1(1-\alpha)}, \end{aligned}$$

where $C_1 = c_0/(1-\alpha)$, one has the desired estimate.

Let

$$M(t) := \max_{0 \leq s \leq t} W(s). \quad (2.21)$$

Note that the function $t \mapsto M(t)$ is monotone increasing. Under these preparations one can prove

Lemma 2.6 *Let $\alpha \in [0, 1)$. Then the following bound holds*

$$\int_{\mathbf{R}^N} wv^2 dx \leq \int_{\mathbf{R}^N} wv^2 dx|_{t=t_0} + C \left(E(v_t, \nabla v)(t_0) + M(t)^{\frac{p+1}{2}} + \int_{t_0}^t s^{-1-\gamma} W(s)^{\frac{p+1}{2}} ds \right) \cdot t^\alpha,$$

for $t \in [t_0, T_m)$ with large $t_0 > 0$.

Proof. It follows from (2.20) that

$$\begin{aligned} &\frac{1}{4} \int_{\mathbf{R}^N} V(x)wv^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^N} wv^2 dx \\ &\leq E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m). \end{aligned}$$

Now by using (2.3) and (2.21) we get

$$\begin{aligned} &\frac{1}{2} \frac{V_0}{(1+t+R)^\alpha} \int_{\mathbf{R}^N} wv^2 dx + \frac{d}{dt} \int_{\mathbf{R}^N} wv^2 dx \\ &\leq 2E(v_t, \nabla v)(t_0) + CM(t)^{\frac{p+1}{2}} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m). \end{aligned}$$

Since the function

$$t \mapsto CM(t)^{\frac{p+1}{2}} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds$$

is monotone increasing, we can apply Lemma 2.5 with

$$h(t) = \int_{\mathbf{R}^N} wv^2 dx, \quad E_0 = 2E(v_t, \nabla v)(t_0), \quad a = 1, \quad c_0 = \frac{V_0}{2},$$

$$f(t) = C(M(t)^{\frac{p+1}{2}} + \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds),$$

and obtain the desired estimate.

Denote by $E_u(t) = \frac{1}{2}(\|u_t(t, \cdot)\|^2 + \|\nabla u(t, \cdot)\|^2)$, where $u \in X_1(0, T_m)$ is the weak solution to problem (1.1)-(1.2). We need the following lemma.

Lemma 2.7 *For each $t \in [0, T_m)$ it is true that*

$$\int_{\mathbf{R}^N} w(t, x)v(t, x)^2 dx \leq C_R(t)\|\nabla u(t, \cdot)\|^2,$$

$$E(v_t, \nabla v)(t) \leq C_R(t)E_u(t),$$

with some t -dependent constant $C_R(t)$ satisfying $\lim_{t \rightarrow +\infty} C_R(t) = +\infty$.

The proof is omitted since it elementary follows from the fact that $v = \frac{u}{w}$, $w = t^{-m}e^{-m_1\phi(x)/t}$ and the compact support of the data.

The standard energy identity associated with the problem (1.1)-(1.2) gives

$$E_u(t) \leq E_u(0) + \frac{1}{p+1}\|u(t, \cdot)\|_{p+1}^{p+1}. \quad (2.22)$$

Then we have the following Lemma.

Lemma 2.8 *Let $t_0 > 0$ be the time defined in Lemmas 2.1-2.6. Then, there exists $T \in (t_0, T_m)$, which depends on $\varepsilon > 0$, such that for all $t \in [0, T]$*

$$E_u(t) \leq 2E_u(0) = (\|u_1\|^2 + \|\nabla u_0\|^2)\varepsilon^2,$$

$$\lim_{\varepsilon \rightarrow +0} T = +\infty.$$

Proof. By using Gagliardo-Nirenberg and Poincaré inequalities, the finite propagation speed and (2.22) we get

$$E_u(t) \leq E_u(0) + C\|\nabla u(t, \cdot)\|^{\theta(p+1)}\|u(t, \cdot)\|^{(1-\theta)(p+1)}$$

$$\leq E_u(0) + C(1+t+R)^{(1-\theta)(p+1)}\|\nabla u(t, \cdot)\|^{p+1}.$$

Thus we have

$$E_u(t) \leq E_u(0) + C(t+R)^{(1-\theta)(p+1)}E_u(t)^{\frac{p+1}{2}}, \quad (2.23)$$

where $\theta = \frac{N(p-1)}{2(p+1)}$, and the constant $C > 0$.

Denote by T the first time $T > 0$ such that $E_u(T) = 2E_u(0)$. Since $E_u(0) < 2E_u(0)$, then $E_u(t) < 2E_u(0)$ for all $t \in [0, T)$. From (2.23) with $t = T$ we have

$$2E_u(0) \leq E_u(0) + C(T + R)^{(1-\theta)(p+1)}(2E_u(0))^{\frac{p+1}{2}}.$$

By solving this inequality with respect to T we find that the time T has the lower bound by $\varepsilon > 0$:

$$\begin{aligned} T &\geq CE_u(0)^{-\frac{p-1}{2(1-\theta)(p+1)}} - R \\ &= C(\varepsilon^2)^{-\frac{p-1}{2(1-\theta)(p+1)}} - R. \end{aligned}$$

This implies that by taking $\varepsilon > 0$ sufficiently small we can make a desired relation $t_0 < T < T_m$, where T_m is the life span of the solution. Note that the T depends only on ε and $T \rightarrow \infty$ when $\varepsilon \rightarrow 0$.

Under these preparations, we go to the final stage of the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.8 we consider the case $t_0 < T < T_m$. For any $\sigma > 0$, since

$$|2wv_t v| \leq \sigma t^\alpha w v_t^2 + \sigma^{-1} t^{-\alpha} w v^2,$$

we have

$$2wv_t v \geq -\sigma t^\alpha w v_t^2 - \sigma^{-1} t^{-\alpha} w v^2.$$

Thus from (2.19) we get

$$\begin{aligned} &\int_{\mathbf{R}^N} P(v_t^2 + |\nabla v|^2) dx - \int_{\mathbf{R}^N} \sigma t^\alpha w v_t^2 dx - \sigma^{-1} t^{-\alpha} \int_{\mathbf{R}^N} w v^2 dx + \int_{\mathbf{R}^N} (w_t + Vw) v^2 dx \\ &\leq 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds. \end{aligned} \quad (2.24)$$

This implies

$$\begin{aligned} &\int_{\mathbf{R}^N} (P - \sigma t^\alpha w)(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^N} (w_t + Vw - \sigma^{-1} t^{-\alpha} w) v^2 dx \\ &\leq 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m). \end{aligned}$$

Because of Lemma 2.1 one has

$$\begin{aligned} &2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds \\ &\geq \int_{\mathbf{R}^N} P(1 - \sigma C_1)(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^N} (Vw + (w_t - \sigma^{-1} t^{-\alpha} w)) v^2 dx \\ &\geq (1 - \sigma C_1) \int_{\mathbf{R}^N} P(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^N} Vw v^2 dx - (C_1 + \sigma^{-1}) t^{-\alpha} \int_{\mathbf{R}^N} w v^2 dx, \quad t \in [t_0, T_m). \end{aligned}$$

This together with Lemma 2.6 yields

$$\begin{aligned} &(1 - \sigma C_1) \int_{\mathbf{R}^N} P(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^N} Vw v^2 dx \\ &\leq 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds \end{aligned}$$

$$\begin{aligned}
& +C(C_1 + \sigma^{-1})t^{-\alpha} \left\{ \int_{\mathbf{R}^N} wv^2 dx|_{t=t_0} + t^\alpha (E(v_t, \nabla v)(t_0) + M(t)^{(p+1)/2} + \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds) \right\} \\
& \leq 2E(v_t, \nabla v)(t_0) + Ct^{-\alpha} \int_{\mathbf{R}^N} wv^2 dx|_{t=t_0} + CE(v_t, \nabla v)(t_0) \\
& \quad + CM(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m),
\end{aligned}$$

where $C = C_\sigma > 0$ is a constant (for the definition of $M(t)$, see (2.21)). By taking $\sigma > 0$ sufficiently small we have

$$\begin{aligned}
& \int_{\mathbf{R}^N} P(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R}^N} V w v^2 dx \\
& \leq C \left(E(v_t, \nabla v)(t_0) + \int_{\mathbf{R}^N} wv^2 dx|_{t=t_0} \right) + CM(t)^{(p+1)/2} + C \int_{t_0}^t s^{-1-\gamma} W(s)^{(p+1)/2} ds,
\end{aligned}$$

for all $t \in [t_0, T_m)$. This estimate together with Lemma 2.7 gives the following crucial bound:

$$W(t) \leq C_{t_0} E_u(t_0) + CM(t)^{(p+1)/2} + \frac{t_0^{-\gamma}}{\gamma} \left(\max_{0 \leq s \leq t} W(s) \right)^{(p+1)/2}, \quad t \in [t_0, T_m).$$

Then from Lemma 2.8 (since $t_0 < T$) we have

$$M(t) \leq C_{t_0} \varepsilon^2 + CM(t)^{(p+1)/2}, \quad t \in [t_0, T_m).$$

By using this estimate and standard arguments as in [36] upon possible additional decreasing of $\varepsilon > 0$, we get

$$M(t) \leq C_{t_0} \varepsilon^2, \quad t \in [t_0, T_m). \quad (2.25)$$

This implies $T_m = +\infty$.

Finally, we derive the decay estimates in Theorem 1.1.

Indeed, from (2.25) we have

$$\int_{\mathbf{R}^N} P(t, x)(v_t(t, x)^2 + |\nabla v(t, x)|^2) dx + \int_{\mathbf{R}^N} V(x)w(t, x)v(t, x)^2 dx \leq C\varepsilon^2, \quad t \geq t_0. \quad (2.26)$$

It follows from the definition of $v = uw^{-1}$ that the second term of the left-hand side of (2.26) satisfies the estimate

$$\int_{\mathbf{R}^N} V(x)w(t, x)^{-1}u(t, x)^2 dx \leq t^m \int e^{\frac{(m-\delta)\phi(x)}{t}} V(x)u^2 dx \leq C\varepsilon^2. \quad (2.27)$$

Further by using the bounds for $\phi(x)$, namely

$$\phi_0(1 + |x|)^{2-\alpha} \leq \phi(x) \leq \phi_1(1 + |x|)^{2-\alpha}, \quad \text{for } x \in \mathbf{R}^N$$

and (2.3) we get the estimate

$$\begin{aligned}
V(x) & \geq C(\phi(x))^{-\frac{\alpha}{2-\alpha}} \\
& = Ct^{-\frac{\alpha}{2-\alpha}} \left(\frac{\phi(x)}{t} \right)^{-\frac{\alpha}{2-\alpha}} \\
& \geq Ct^{-\frac{\alpha}{2-\alpha}} e^{-\delta \frac{\phi(x)}{t}},
\end{aligned} \quad (2.28)$$

where $C > 0$ and $t \geq t_0$ is sufficiently large. We complete the decay estimate for L^2 - norm of solution u by substituting this lower bound for $V(x)$ into inequality (2.27).

Next, we derive the energy decay estimate for the solution u of (1.1)-(1.2).

Notice that

$$v_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \geq \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2 u^2,$$

and

$$|\nabla v|^2 = |-w^{-2}u\nabla w + w^{-1}\nabla u|^2 \geq \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2 u^2.$$

Thus we have

$$\frac{1}{2}Pw^{-2}(u_t^2 + |\nabla u|^2) \leq P(v_t^2 + |\nabla v|^2) + 3Pw^{-4}(w_t^2 + |\nabla w|^2)u^2.$$

Integrating this inequality over \mathbf{R}^N , and applying (2.26) one obtains

$$\frac{1}{2} \int_{\mathbf{R}^N} Pw^{-2}(u_t^2 + |\nabla u|^2) dx \leq C\varepsilon^2 + 3 \int_{\mathbf{R}^N} Pw^{-4}(w_t^2 + |\nabla w|^2)u^2 dx. \quad (2.29)$$

It is easy to check that P and w satisfy

$$Pw^{-3}(w_t^2 + |\nabla w|^2) \leq CV(x).$$

By using the above inequality and estimates (2.29) and (2.27) we get

$$\frac{1}{2} \int_{\mathbf{R}^N} Pw^{-2}(u_t^2 + |\nabla u|^2) dx \leq C\varepsilon^2 + C \int_{\mathbf{R}^N} w^{-1}V(x)u^2 dx \leq C\varepsilon^2.$$

This implies the energy decay estimate in Theorem 1.1.

Finally, from (2.25) and Lemma 2.3 we find that

$$\int_{\mathbf{R}^N} \left(\frac{6}{t} + \frac{S(x)}{t^2}\right)^{-1} w^{-1} |u|^{p+1} dx \leq Ct^{-\gamma},$$

so that the decay estimate for the L^{p+1} norm as written in Theorem 1.1 holds.

3 Blow-up

In this section we prove the blow-up part of Theorem 1.2. Recall that the power of nonlinearity $|u|^p$ is subcritical or critical, i.e. $p \leq p_c(N, \alpha)$. We rely on the method of test functions developed by Zhang [37], [38], [39].

Proof of Theorem 1.2. First we find a non-negative $\phi \in C_0^\infty(\mathbf{R} \times \mathbf{R}^N)$, such that

$$\phi(t, x) = \begin{cases} 1, & \text{if } (t, x) \in [-1, 1] \times B(R), \\ 0, & \text{if } (t, x) \in (\mathbf{R} \times \mathbf{R}^N) \setminus ([-2, 2] \times B(2R)). \end{cases}$$

We can also satisfy the additional condition

$$|D^2\phi(t, x)|^4 + |D\phi(t, x)|^2 \leq C\phi(t, x), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^N,$$

where $D = (\partial_t, \nabla)$ and $C > 0$ is some constant; see [39] for the existence of such functions. Then the test function ϕ_T is defined by

$$\phi_T(t, x) = \phi\left(\frac{t}{T^{2-\alpha}}, \frac{x}{T}\right), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^N,$$

with some large parameter T . Let P_T be the subset of $\mathbf{R} \times \mathbf{R}^N$ where $\phi_T = 1$ and

$$Q_T = (\text{supp}(D^2\phi_T) \cup \text{supp}(D\phi_T)) \cap ([0, +\infty) \times \mathbf{R}^N),$$

i.e., Q_T is the support of derivatives restricted to $t \geq 0$. It is easy to see that

$$\begin{aligned} P_T &\supset \{(t, x) : t \leq T^{2-\alpha} \text{ and } |x| \leq RT\}, \\ Q_T &\subset \{(t, x) : t > T^{2-\alpha} \text{ or } |x| > RT\}. \end{aligned} \quad (3.1)$$

Assume that a global solution u exists (i.e., $T_m = +\infty$) when

$$1 < p \leq p_c(N, \alpha) = 1 + 2/(N - \alpha),$$

and

$$\int_{\mathbf{R}^N} (V(x)u_0(x) + u_1(x))dx > 0.$$

To obtain a contradiction, we multiply the equation (1.1) by ϕ_T^q , with $q = 2p/(p-1)$, and integrate by parts over $[0, +\infty) \times \mathbf{R}^N$:

$$\begin{aligned} \int_0^\infty \int_{\mathbf{R}^N} u(\partial_t^2 \phi_T^q - \Delta \phi_T^q - V \partial_t \phi_T^q) dx dt &= \int_0^\infty \int_{\mathbf{R}^N} |u|^p \phi_T^q dx dt \\ &\quad + \int_{\mathbf{R}^N} (Vu_0 + u_1) dx. \end{aligned} \quad (3.2)$$

Here we use $\phi_T(0, x) = 1$, $\partial_t \phi_T(0, x) = 0$, and the initial conditions on u to evaluate boundary integrals at $t = 0$.

Next, we estimate the integral on the left by Hölder's inequality and compare it with the integral on the right side. A straightforward calculation yields

$$\begin{aligned} |\partial_t^2 \phi_T^q - \Delta \phi_T^q - V \partial_t \phi_T^q| &\leq C(\phi_T^{q-1} |D^2 \phi_T| + \phi_T^{q-2} |D \phi_T|^2 + V \phi_T^{q-1} |\partial_t \phi_T|) \\ &\leq C(T^{2\alpha-4} \phi_T^{q-3/4} + T^{-2} \phi_T^{q-3/4}) \\ &\quad + C(T^{-2} \phi_T^{q-1} + T^{2\alpha-4} \phi_T^{q-1} + T^{\alpha-2} V \phi_T^{q-1/2}) \end{aligned}$$

for $(t, x) \in Q_T$, while $\partial_t^2 \phi_T^q - \Delta \phi_T^q - V \partial_t \phi_T^q$ is identically 0 for $(t, x) \notin Q_T$. Since $\alpha \in [0, 1)$ and $\phi_T \leq C$, the expression is bounded by $C(T^{-2} \phi_T^{q-1} + T^{\alpha-2} V \phi_T^{q-1/2})$. Thus, the left side of identity (3.2) satisfies

$$\begin{aligned} &\left| \int_0^\infty \int_{\mathbf{R}^N} u(\partial_t^2 \phi_T^q - \Delta \phi_T^q - V \partial_t \phi_T^q) dx dt \right| \\ &\leq C \int_0^\infty \int_{Q_T} |u| (T^{-2} \phi_T^{q-1} + T^{\alpha-2} V \phi_T^{q-1/2}) dx dt \\ &\leq C \left(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q dx dt \right)^{1/p} I^{(p-1)/p}(T), \end{aligned} \quad (3.3)$$

with

$$\begin{aligned} I(T) &= T^{-2p/(p-1)} \int_0^\infty \int_{\mathbf{R}^N} \phi_T^{q-p/(p-1)} dx dt \\ &\quad + T^{(\alpha-2)p/(p-1)} \int_0^\infty \int_{\mathbf{R}^N} V^{p/(p-1)} \phi_T^{q-p/(2(p-1))} dx dt. \end{aligned}$$

Notice that in the above identity all exponent of ϕ_T are positive since $V(x) = V_0(1 + |x|)^{-\alpha}$ satisfies the condition **(A)**. Hence

$$\begin{aligned} I(T) &\leq CT^{-2p/(p-1)} \int_0^{2T^{2-\alpha}} \int_{|x| \leq 2RT} dxdt \\ &\quad + CT^{(\alpha-2)p/(p-1)} \int_0^{2T^{2-\alpha}} \int_{|x| \leq 2RT} (1 + |x|)^{-\alpha p/(p-1)} dxdt, \end{aligned}$$

which readily simplifies to

$$I(T) \leq C \begin{cases} T^{-2p/(p-1)+2-\alpha+N} + T^{(\alpha-2)p/(p-1)+2-\alpha}, & \text{if } \alpha p/(p-1) > N, \\ T^{-2p/(p-1)+2-\alpha+N} + T^{(\alpha-2)p/(p-1)+2-\alpha} \log T, & \text{if } \alpha p/(p-1) = N, \\ T^{-2p/(p-1)+2-\alpha+N} + T^{-2p/(p-1)+2-\alpha+N}, & \text{if } \alpha p/(p-1) < N. \end{cases}$$

An upper bound is $I(T) \leq CT^{-2p/(p-1)+2-\alpha+N}$. We can return to (3.3) and write

$$\begin{aligned} &\left| \int_0^\infty \int_{\mathbf{R}^N} u(\partial_t^2 \phi_T^q - \Delta \phi_T^q - V \partial_t \phi_T^q) dxdt \right| \\ &\leq C \left(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q dxdt \right)^{1/p} T^{-2+(2-\alpha+N)(p-1)/p}, \end{aligned}$$

where C is independent of T . Substituting this estimate into (3.2), we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^N} |u|^p \phi_T^q dxdt \\ &\leq C \left(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q dxdt \right)^{1/p} T^{-2+(2-\alpha+N)(p-1)/p}. \end{aligned} \quad (3.4)$$

Finally, we show that the above inequality can not hold as $T \rightarrow \infty$. If $p \leq p_c(N, \alpha)$, then

$$-2 + (2 - \alpha + N)(p - 1)/p \leq 0,$$

and (3.4) shows that

$$\left(\int_0^\infty \int_{P_T} |u|^p dxdt \right)^{(p-1)/p} \leq C.$$

Letting $T \rightarrow \infty$ and using (3.1), we conclude that $u \in L^p([0, +\infty) \times \mathbf{R}^N)$. Hence (3.1) also implies that $\|u\|_{L^p(Q_T)} \rightarrow 0$ as $T \rightarrow \infty$. Passing to the limit in (3.4), we obtain $\|u\|_{L^p([0, +\infty) \times \mathbf{R}^N)} \leq 0$ for any $1 < p \leq p_c(N, \alpha)$. This is impossible, since u is a non-trivial solution.

Acknowledgement. The work of the first author was supported in part by Grant-in-Aid for Scientific Research (C) 19540184 of JSPS.

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