Name: ____________________________________

Math 110 – Exam 2

20 February 2004

Directions: There are nine questions on this exam. Answer every question. Show all work and justify your answers. Each question is worth five points.

1. (a) Who was Georg Cantor?

   (b) What did he prove with the *diagonalization proof*?

2. Explain how (without counting) you would decide if Imelda Marcos had the same number of right shoes as left shoes.

3. (a) Give a completely accurate definition of *one-to-one correspondence*.

   (b) When do two collections of objects have the same *cardinality*?

4. Consider the set of people living in the United States and the set of phone numbers in service. Is the pairing which matches people to their phone numbers a one-to-one correspondence? Why or why not?

5. Consider the 52 cards in a standard deck of playing cards (no jokers). Is the pairing which matches the red cards to the black cards a one-to-one correspondence? Why or why not?

6. (a) Are the sets $\{1, 2, 3, 4, 5, \ldots\}$ and $\{21, 22, 23, 24, 25, \ldots\}$ equally numerous? Why or why not?

   (b) What about the sets $\{1, 2, 3, 4, 5, \ldots\}$ and $\{1, 8, 27, 64, 125, 216, 343, 512, \ldots\}$? Why or why not?

7. (a) Are the sets $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ and $\{1, 2, 3, 4, \ldots\}$ equally numerous? (These sets are also known as the *integers* and the *natural numbers*.) Why or why not?

   (b) Does the set of rational numbers have the same cardinality as the set of natural numbers? Very briefly explain your answer (you don't need to give a complete proof).

8. The Hilbert Hotel has an infinite number of rooms — one for each natural number. The hotel is full. An infinite number of people arrive, wanting to stay at the hotel. Is there a way for the manager to give each person his own room (without kicking anyone out)? How can it be done or why is it impossible? (There is more than one right answer.)

9. Dover says that he has found a one-to-one correspondence between the set of natural numbers and the set of real numbers. His pairing begins like this:

$$
\begin{array}{ccl}
1 & \leftrightarrow & 1.23456789101112\ldots \\
2 & \leftrightarrow & 9.87654321012345\ldots \\
3 & \leftrightarrow & 7.23223222322223\ldots \\
4 & \leftrightarrow & 0.76567876545678\ldots \\
5 & \leftrightarrow & 0.99999999999999\ldots \\
 & \vdots &
\end{array}
$$

Use the method described in class and in the textbook to come up with the first few decimal places of a number that you are absolutely certain is not on Dover's list.

[Bonus: 3 points] Suppose the Cardinals (the infinite baseball team) wore uniforms labeled: 1, 2, 3, 4, 5, ... (one for each natural number). They are standing in line by uniform number. Each player has a penny, and you ask the players to flip their pennies at the same time. You ask them to shout out (in order) what they flipped (H for heads or T or tails). Consider the set of all possible outcomes of their flipping (all possible sequences of H and T). Does the set of possible outcomes have the same cardinality as the natural numbers or not? Justify your answer.