

# QUIZ VI, MATH 251, 1 DECEMBER, 2009

NAME: Solutions

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1. (15 points): Let

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of  $A$ .

**Solution:** The eigenvalues are  $\lambda = 3$  and  $\lambda = -1$ , since the matrix is lower triangular. (If you found these using the characteristic polynomial then you did more work than needed. But all that matters here is the solution.) Now consider

$$\lambda I - A = \begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix}.$$

To find the eigenvectors we seek the nontrivial solutions of  $(\lambda I - A)\mathbf{x} = \mathbf{0}$ .

( $\lambda = 3$ ):

$$[\lambda I - A|\mathbf{x}] = \left[ \begin{array}{cc|c} 0 & 0 & 0 \\ -8 & 4 & 0 \end{array} \right].$$

The solution of this system is

$$\mathbf{x} = \begin{bmatrix} \frac{s}{2} \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

To pick one eigenvector we can set  $s = 1$ . (In fact, you can set  $s$  equal to any number except  $s = 0$ , which would yield a trivial solution.)

( $\lambda = -1$ ):

$$[\lambda I - A|\mathbf{x}] = \left[ \begin{array}{cc|c} -4 & 0 & 0 \\ -8 & 0 & 0 \end{array} \right].$$

The solution of this system is

$$\mathbf{x} = \begin{bmatrix} 0 \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To pick one eigenvector we can set  $s = 1$  as before.

2. (15 points): Let

$$A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find  $\text{rank}(A)$ . (b) Find a basis for the column space of  $A$ . (c) Find a basis for the row space of  $A$ . (d) Find  $\text{nullity}(A)$  without finding a basis for the null space of  $A$ .

**Solution:** These questions are easily answered since  $A$  is in row echelon form already. (a)  $\text{rank}(A)$  is the number of leading ones. Hence  $\text{rank}(A) = 3$ .

(b) A basis for the column space is give by

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} \right\}.$$

(c) A basis for the row space is give by

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

where we have rewritten the row vectors are column vectors.

(d) By the rank plus nullity theorem,  $\text{nullity}(A)$  is the number of columns of  $A$  minus  $\text{rank}(A)$ . Hence  $\text{nullity}(A) = 1$ , since there are 4 columns.