Practice Final Exam, Math 251
3 December, 2009

Name:

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1. Let
$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$
Find the eigenvalues and eigenvectors of A.

2. Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix}.$$
(**a**) Find the eigenvalues and eigenvectors of A. (**b**) Write down a basis for the null space of A.

3. Let
$$A = \begin{bmatrix} 1 & 0 & 3 & 2 & -1 \\ 0 & 1 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
(**a**) Find rank(A). (**b**) Find a basis for the column space of A. (**c**) Find a basis for the row space of A. (**d**) Find nullity(A) without finding a basis for the null space of A.

4. (**a**) Find the general solution of the system represented by the following augmented matrix:
$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$
(**b**) Write down a basis for the null space of the coefficient matrix.

5. Consider the matrices
$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
where R is row equivalent to A. (**a**) Find rank(A). (**b**) Find a basis for the column space of A. (**c**) Find a basis for the row space of A. (**d**) Find nullity(A) without finding a basis for the null space of A.

6. Suppose $k \neq 0$ and

$$A = \begin{bmatrix} k & 0 & 0 \\ 1 & k & 0 \\ 0 & 1 & k \end{bmatrix}.$$

Find A^{-1}.

7. Let

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Find $\det(A)$.

8. Let $A \in \mathbb{R}^{3\times 3}$, and $\mathbf{b} \in \mathbb{R}^3$. Suppose $\text{rank}(A) = 2$ and $\text{rank}\left([A|\mathbf{b}]\right) = 3$. (**a**) Is $A\mathbf{x} = \mathbf{b}$ consistent? If so what are the number of parameters in the general solution? (**b**) Find nullity(A). (**c**) What are the number of parameters in the general solution to $A\mathbf{x} = \mathbf{0}$?

9. Suppose that $A \in \mathbb{R}^{n\times n}$ is invertible. Use the determinant function to prove that $A^{-1}A^TA$ is also invertible.

10. Let $A \in \mathbb{R}^{5\times 9}$, and $\mathbf{b} \in \mathbb{R}^5$. Suppose $\text{rank}(A) = 2$ and $\text{rank}\left([A|\mathbf{b}]\right) = 2$. (**a**) Is $A\mathbf{x} = \mathbf{b}$ consistent? If so what are the number of parameters in the general solution? (**b**) Find nullity(A). (**c**) What are the number of parameters in the general solution to $A\mathbf{x} = \mathbf{0}$?

11. Suppose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and $\det(A) = 10$. (**a**) Calculate

$$\det\left(\begin{bmatrix} g & h & i \\ 3d & 3e & 3f \\ a & b & c \end{bmatrix}\right).$$

(**b**) Is A invertible? If so, calculate $\det((10A)^{-1})$.

12. In the following system of equations, find conditions that the b's must satisfy for the system to be consistent.

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 &= b_1 \\ x_1 - x_2 - x_3 &= b_2 \\ -x_1 + x_2 - x_3 &= b_3 \end{aligned}$$

13. Suppose the augmented matrix $[A|\mathbf{b}]$ is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & b_1 \\ 0 & 1 & -5 & b_1 - 2b_2 \\ 0 & 0 & 0 & 2b_1 - b_2 + 3b_3 \end{array}\right].$$

Find all the vectors $\mathbf{b} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

14. Suppose

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Is $\mathbf{b}$ in the column space of A? If so, express $\mathbf{b}$ as a linear combination of the column vectors of A.

15. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Find a subset of $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ that forms a basis for span(S). What is span(S)?

16. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a subset of vector from $\mathbb{R}^3$. Is it possible that S is a basis for $\mathbb{R}^3$? Why or why not?

17. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a subset of vector from $\mathbb{R}^5$. Is it possible that S is a basis for $\mathbb{R}^5$? Why or why not?

18. Fix $\mathbf{x} \in \mathbb{R}^n$. Consider the set of all $n \times n$ matrices A that have the property $A\mathbf{x} = \mathbf{0}$. Show that, with the usual operations of matrices, this is a vector subspace of the $n \times n$ matrices.

19. Fix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Consider the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ that have the property $A\mathbf{x} = \mathbf{0}$. Show that, with the usual operations of vector addition and scalar multiplication, this is a vector subspace of $\mathbb{R}^n$. What is the name of this vector subspace?

20. Consider the subset of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$$

with the standard matrix-matrix addition and scalar-matrix multiplication. Show that this a vector subspace of the 2×2 matrices.

21. Let E, F, $S \in \mathbb{R}^{n\times n}$. Suppose S is invertible and $E = S^{-1}FS$. Show that $\det(E) = \det(F)$.

22. Consider subset of $\mathbb{R}^3$ consisting of all triples of the form $(0, y, z)$ together with the usual operations of vector addition and scalar-vector multiplication. Show that this is a vector subspace of $\mathbb{R}^3$.

23. (**a**) Let $\mathbf{u} = (4, k, k)$ and $\mathbf{v} = (1, k, -5)$. For what value (or values) of k are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? (**b**) Setting k equal to one of those values (for which $\mathbf{u}$ and $\mathbf{v}$ orthogonal), confirm the Pythagorean theorem holds: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

24. Let $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, -1, 2)$, and $\mathbf{v}_3 = (4, 1, -1)$. Is S linearly independent or linearly dependent?

25. Consider the set of all vectors in $\mathbb{R}^4$ of the form $(a, a-b, c, c-b)$, where a, b, and c are real numbers. What is the dimension of this vector subspace of $\mathbb{R}^4$?