

EXAM II, MATH 251, 22 OCTOBER, 2009

NAME: Solutions

Instructions: Put away all electronic devices, calculators, cell phones, *et cetera*. Write only on the paper provided. Please box the final result of your calculations.

1. **(20 points):** Let $E, F, S \in \mathbb{R}^{n \times n}$. Suppose S is invertible and $E = S^{-1}FS$. Show that $\det(E) = \det(F)$.

Proof: Using the property $\det(AB) = \det(A)\det(B)$, we find

$$\det(E) = \det(S^{-1}FS) = \det(S^{-1})\det(FS) = \det(S^{-1})\det(F)\det(S) .$$

Using the property $\det(S^{-1}) = \frac{1}{\det(S)}$, we have

$$\det(E) = \frac{1}{\det(S)} \det(F)\det(S) = \det(F)$$

and we are done. ///

2. **(20 points):** Suppose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and $\det(A) = -5$. **(a)** Calculate

$$\det \left(\begin{bmatrix} d & a & g \\ e & b & h \\ f & c & i \end{bmatrix} \right).$$

(b) Is A invertible? If so, calculate $\det((2A)^{-1})$.

Solution: **(a)**

$$\det(A) = \det(A^T) = \det \left(\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \right) = -\det \left(\begin{bmatrix} d & a & g \\ e & b & h \\ f & c & i \end{bmatrix} \right),$$

where we performed a single column interchange to obtain the last matrix. Thus

$$\det \left(\begin{bmatrix} d & a & g \\ e & b & h \\ f & c & i \end{bmatrix} \right) = 5.$$

(b) A is invertible because the determinant is non-zero. Now

$$\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{2^3 \det(A)} = \frac{1}{8(-5)} = -\frac{1}{40}.$$

3. **(20 points):** (a) What is the standard matrix for the reflection transformation about the $x - y$ plane in \mathbb{R}^3 ? Use matrix multiplication to find the reflection of the point $(-2, 1, 7)$ about the $x - y$ plane. (b) What is the standard matrix for the reflection transformation about the $y - z$ plane in \mathbb{R}^3 ? Use matrix multiplication to find the reflection of the point $(6, -1, 5)$ about the $y - z$ plane.

Solution: (a) The standard matrix is the coefficient matrix of the following system of equations:

$$\begin{aligned}w_1 &= x + 0y + 0z \\w_2 &= 0x + y + 0z \ . \\w_3 &= 0x + 0y - z\end{aligned}$$

Thus

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ .$$

Reflecting the point $(-2, 1, 7)$ we simply multiply to find

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -7 \end{bmatrix} \ .$$

(b) The standard matrix is the coefficient matrix of the following system of equations:

$$\begin{aligned}w_1 &= -1x + 0y + 0z \\w_2 &= 0x + y + 0z \ . \\w_3 &= 0x + 0y + z\end{aligned}$$

Thus

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

Reflecting the point $(6, -1, 5)$ we simply multiply to find

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ 5 \end{bmatrix} \ .$$

4. **(20 points):** (a) Let $\mathbf{u} = (3k, 1, k)$ and $\mathbf{v} = (-1, 2, k)$. For what value (or values) of k are \mathbf{u} and \mathbf{v} orthogonal? (b) Setting k equal to one of those values (for which \mathbf{u} and \mathbf{v} orthogonal), confirm the Pythagorean theorem holds: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Solution: (a) \mathbf{u} and \mathbf{v} are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. Thus we calculate

$$0 = \mathbf{u} \cdot \mathbf{v} = (3k)(-1) + (1)(2) + (k)(k) = k^2 - 3k + 2 = (k - 1)(k - 2) .$$

Thus \mathbf{u} and \mathbf{v} are orthogonal if and only if $k = 1$ or $k = 2$.

(b) Taking $k = 1$, we find $\mathbf{u} = (3, 1, 1)$ and $\mathbf{v} = (-1, 2, 1)$. Hence, $\mathbf{u} + \mathbf{v} = (2, 3, 2)$. Calculating the norms we find

$$\begin{aligned} \|\mathbf{u}\|^2 &= (3)^2 + (1)^2 + (1)^2 = 11 , \\ \|\mathbf{v}\|^2 &= (-1)^2 + (2)^2 + (1)^2 = 6 , \\ \|\mathbf{u} + \mathbf{v}\|^2 &= (2)^2 + (3)^2 + (2)^2 = 17 . \end{aligned}$$

The Pythagorean theorem is confirmed: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

5. (20 points): Without directly evaluating, show that

$$\det \left(\begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} \right) = 0 .$$

Solution: Notice that

$$\begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2+R_1 \rightarrow R_1} \begin{bmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} .$$

The matrix on the right has the same determinant as that on the left. But the matrix on the right has determinant equal to zero, since it has two proportional rows.

6. **(20 points):** In the following system of equations, find conditions that the b 's must satisfy for the system to be consistent.

$$\begin{aligned} 6x_1 - 4x_2 + 2x_3 &= b_1 \\ 3x_1 - 2x_2 - x_3 &= b_2 \\ -3x_1 + 2x_2 - x_3 &= b_3 \end{aligned}$$

Solution: Using Gaussian Elimination

$$\begin{aligned} \left[\begin{array}{ccc|c} 6 & -4 & 2 & b_1 \\ 3 & -2 & -1 & b_2 \\ -3 & 2 & -1 & b_3 \end{array} \right] & \xrightarrow{\frac{1}{6}R_1 \rightarrow R_1} & \left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & b_1/6 \\ 3 & -2 & -1 & b_2 \\ -3 & 2 & -1 & b_3 \end{array} \right] \\ & \xrightarrow{-3R_1 + R_2 \rightarrow R_2} & \left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & b_1/6 \\ 0 & 0 & -2 & b_2 - b_1/2 \\ -3 & 2 & -1 & b_3 \end{array} \right] \\ & \xrightarrow{3R_1 + R_3 \rightarrow R_3} & \left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & b_1/6 \\ 0 & 0 & -2 & b_2 - b_1/2 \\ 0 & 0 & 0 & b_3 + b_1/2 \end{array} \right] \\ & \xrightarrow{-\frac{1}{2}R_2 \rightarrow R_2} & \left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & b_1/6 \\ 0 & 0 & 1 & b_1/4 - b_2/2 \\ 0 & 0 & 0 & b_3 + b_1/2 \end{array} \right] \end{aligned}$$

The system is consistent if $b_3 + b_1/2 = 0$.