Research Statement

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My research has focused on the Loewner equation, which is a topic that relates probability, analysis, and physics. The Loewner equation gives a correspondence between real functions and special sets in the complex plane. My work focuses on understanding this correspondence. In the future, I plan on working on questions inspired by my current work.

Introduction to the Loewner Equation

Charles Loewner first discovered his namesake equation while trying to prove the Bieberbach conjecture around 1923. In 1985, Louis de Branges used it in his proof of the conjecture. Fifteen years later, Oded Schramm was investigating scaling limits of the Loop-erased random walks (LERW) and discovered its relationship to the Loewner equation driven by Brownian motion. His discovery answered questions connected to self-avoiding walks (SAW), the Ising model, percolation, conformal field theory (CFT), diffusion limited aggregation (DLA), and many others. This interconnectivity inspires investigation of the Loewner equation.

The (chordal) Loewner equation is the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z \tag{1}$$

for $z \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, where $\lambda : [0, T] \to \mathbb{R}$ is called the driving function. For $z \in \mathbb{H}$, a solution exists up to a maximum time, call it T_z. The collection of points

$$K_t = \{z \in \mathbb{H} : T_z \leq t\} \tag{2}$$

is called a hull. Each driving function corresponds to a hull and the hulls can be thought of as the points killed at a particular time (since $g_t(z) - \lambda(t) = 0$). The map g_t is a conformal map from $\mathbb{H} \setminus K_t$ to $\mathbb{H}$. It is often possible to define a curve $\gamma : [0, T] \to \overline{\mathbb{H}}$, called the trace of K_t, so that $\mathbb{H} \setminus K_t$ is the unbounded component of γ.

When the Loewner equation is driven by $\lambda(t) = \sqrt{\kappa} B_t$, for a Brownian motion B_t, the resulting family of random hulls is called the Schramm-Loewner Evolution (SLE_κ) in honor of Oded Schramm. For any κ, a.s. the trace γ exists, but some of the family's properties depend on κ. For example, we have the following changes based on κ (called phase transitions):

- $\kappa \in [0, 4] : \gamma(t)$ is a.s. a simple path in $\mathbb{H} \cup \{0\}$
- $\kappa \in (4, 8) : \gamma(t)$ is a.s. a non-simple path
- $\kappa \in [8, \infty)$: $\gamma(t)$ is a.s. a spacefilling curve

One property that is crucial for the phase transitions is that the traces (and hulls) exhibit Brownian scaling (i.e. B_t and cB_{t/c^2} have the same distribution). The phase transitions in this probablistic setting motivate the study in the deterministic setting.

One natural collection of deterministic functions with this type of scaling relationship is Lip(1/2). We say $\lambda : [0, T] \to \mathbb{R}$ is Lip(1/2) if there exists $c \geq 0$ so that $|\lambda(t) - \lambda(s)| \leq c\sqrt{|t-s|}$ for $s, t \in [0, T]$ and the smallest such c is the Lip(1/2) norm of λ, denoted $\|\lambda\|_{1/2}$. The phase transitions here are different than for SLE. Unlike the SLE setting, for any value c, we can find $\lambda \in$Lip(1/2) with $\|\lambda\|_{1/2} = c$ that generates a simple curve (e.g. $\lambda(t) = c\sqrt{t}$). However, we do have the following

- $\|\lambda\|_{1/2} < 4$ generates a simple curve
- $\|\lambda\|_{1/2} < 4.0001$ generates a curve that does not have area

These results are in [Lin05] and [LR12], respectively.

Generalizing slightly, the multiple Loewner equation, which is a weighted sum of Loewner equations, is the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \sum_{j=1}^{n} \frac{2w_j(t)}{g_t(z) - \lambda_j(t)}, \quad g_0(z) = z \tag{3}$$

where $\lambda_1, ..., \lambda_n : [0, T] \to \mathbb{R}$ and $w_1, ..., w_n : [0, T] \to [0, 1]$ so that $\sum_{j=1}^{n} w_j(t) = 1$ for $t \in [0, T]$. Here the hulls are often the union of disjoint hulls from the chordal Loewner equation. Moving from one driving function to multiple driving functions allows for points to be killed off in mutliple locations simultaneously. The multiple Loewner equation is related to Laplacian path models and the study of multiple SLE is also of interest.

Current Work

Multiple Loewner Equation

Kager, Nienhuis, and Kadanoff [KNK04] conjectured that the hull generated by the multiple Loewner equation with two specific driving functions and weights could be realized by a rapidly and randomly oscillating function driving the chordal Loewner equation. In this paper, I prove a more general version of their conjecture. I show that if a collection of driving functions generate disjoint hulls, then for any weights, the hull can be generated from the (chordal) Loewner equation by a function that makes rapid, random jumps between the driving functions.

Proposition 1. *Let K_t be a hull generated from the multiple Loewner equation with continuous driving functions. Then K_t is the limit of hulls generated by a sequence of randomly and rapidly oscillating functions.*

In order to prove their conjecture, I also generalize a result of Roth and Schleissinger. In [RS17], it is shown that if a hull is the union of disjoint slits, then there are continuous driving functions and constant weights that generate the hull. I generalize this to hulls that are the union of disjoint Loewner hulls. A hull that can be generated from the chordal Loewner equation driven by a continuous function is called a Loewner hull. This distinguishes hulls generated from the chordal and multiple Loewner equations.

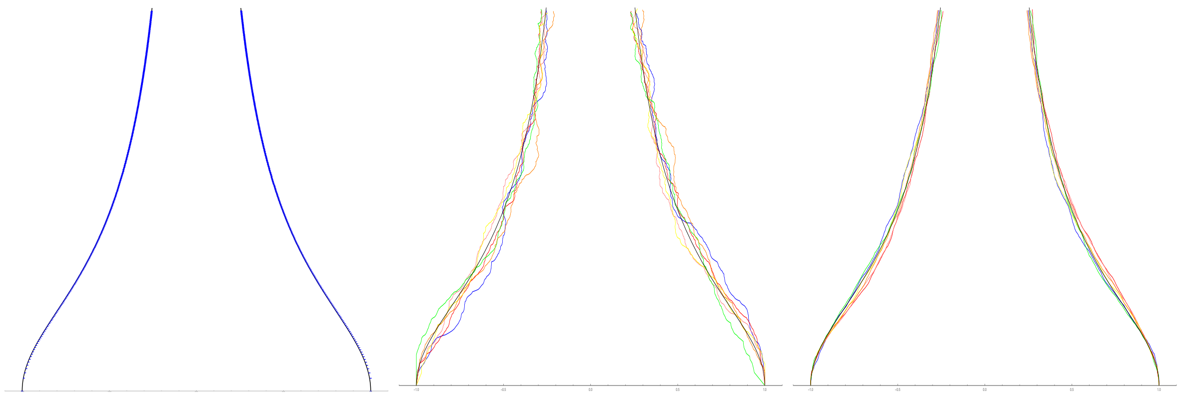

Figure 1: 1000 controlled oscillations (left), 1000 and 10000 random oscillations (center and right, resp)

Theorem 2. *Let $K_1, ..., K_n$ be disjoint Loewner hulls. Let $hcap(K_1 \cup \cdots \cup K_n) = 2T$. Then there exist constants $w_1, ..., w_n \in (0,1)$ with $\sum_{k=1}^n w_k = 1$ and continuous driving functions $\lambda_1, ..., \lambda_n : [0,T] \to \mathbb{R}$ so that*

$$\frac{\partial}{\partial t} g_t(z) = \sum_{k=1}^{n} \frac{2w_k}{g_t(z) - \lambda_k(t)}, \quad g_0(z) = z \tag{4}$$

satisfies $g_T = g_{K_1 \cup \cdots \cup K_n}$.

The key to proving this generalization is by comparing the growth of slits and hulls. In particular, I investigate how the boundary of K_t grows and interacts with the driving function over time. The proof of Proposition 1 is constructive and leads to a simulation method for the hulls generated from the multiple Loewner equation. I have applied this simulation method to the setting of the conjecture. Figure 1 shows three hulls with the computed hull in black and the simulations in other colors. Since we randomly oscillate, the figure has a few different sample oscillations (shown with the different colors). The convergence of the random oscillation appears slower than the convergence in the controlled oscillation, which is expected due to a convergence that occurs with the weights. The important takeaway is that in the random case, if we oscillate enough, the simulation is a great visualization of the curve. These simulation were carried out in Mathematica.

Time-Changing the Loewner Equation (with K. Kobayashi and J. Lind)

If X is a Lévy process and E is a nonnegative, nondecreasing, continuous process independent of X, we say that X_{E_t} is a time-changed process. These processes have been studied by many pure and applied mathematicians. In this paper, we consider the affects of time-changing the driving function of the Loewner equation. Moreover, we show that the behavior of the time-changed hull is often *radically* different than the original hull.

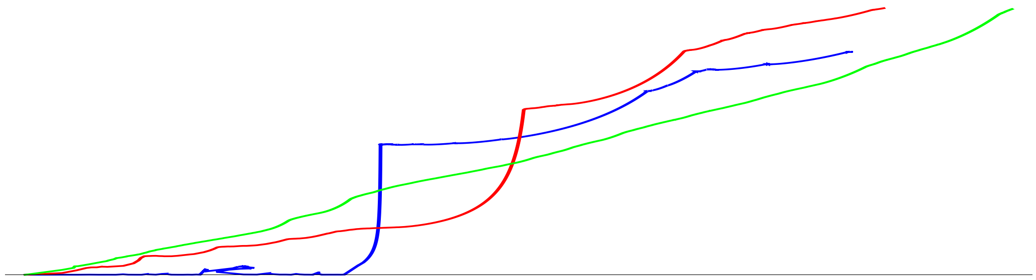

Figure 2: Hulls generated by time-changed square root with varying α values

Before we state the results we will review some needed information. A Lévy process with nondecreasing paths is called a subordinator. Recall that a process X is self-similar of index $\alpha > 0$ if $(X_{ct})_{t\geq 0}$ and $(c^\alpha X_t)_{t\geq 0}$ for $c > 0$ have the same distribution. An α-stable subordinator is a subordinator with self-similar index $1/\alpha$, which means its inverse has self-similar index α. The inverse of an α-stable subordinator is a particulary intriguing time change, which has applications in finance, hydrology, and fractional calculus for example. A key property of the inverse is the fact that it is continuous. Further, Brownian motion time-changed by the inverse of an α-stable subordinator is of great interest because it is a non-Markovian process. Our first result, which is analogous to Proposition 3.2 in [CR09], says that when the time-changed hulls are rescaled appropriately, they simply converge to a vertical slit.

Proposition 3. *Let K_t be the hull driven by X_{E_t}, where X is a continuous, self-similar process of index $H > 0$ and E is a nonnegative, continuous, self-similar process of index $\alpha > 0$, independent of X.*

(a) If $2H\alpha < 1$, then as $r \to \infty$, the rescaled hulls $\frac{1}{r}K_{r^2}$ converge to the vertical line segment $[0, 2i]$ (in the Hausdorff metric) in probability. If $2H\alpha > 1$, then the same conclusion holds as $r \to 0$.

(b) Suppose that the law of X_1 is continuous. If $2H\alpha < 1$, for all $\epsilon > 0$,

$$\lim_{r\to 0} \mathbb{P}\left(\frac{1}{r}K_{r^2} \cap \{y > \epsilon \text{ and } |x| < 1/\epsilon\} \neq \emptyset\right) = 0. \tag{5}$$

If $2H\alpha > 1$, then the same conclusion holds with the limit as $r \to \infty$.

The next result exemplifies another way that the time-change fundamentally alters the hull.

Theorem 4. *Let E_t be the inverse of a subordinator whose Lévy measure is infinite and Laplace exponent ψ is regularly varying at ∞ with index $\alpha \in (0, 1)$. Let ϕ be a deterministic function.*

1. *If $\phi \in$Lip(β) with $\beta > \frac{1}{2\alpha}$, then a.s. $\lambda(t) = \phi(E_t)$ generates a simple curve.*
2. *If there exists $\tau, \epsilon, c > 0$ and $0 < \beta < \frac{1}{2\alpha}$ so that $|\phi(\tau) - \phi(t)| \geq c(\tau - t)^\beta$ for all $t \in (\tau - \epsilon, \tau)$, then a.s. $\lambda(t) = \phi(E_t)$ does not generate a simple curve.*

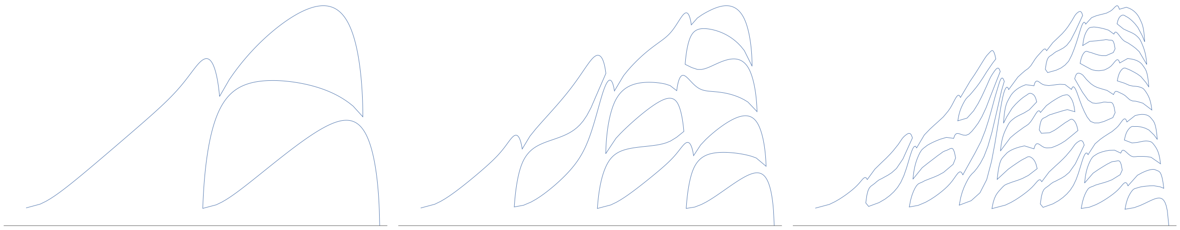

Figure 3: Recursively generated hulls for $c = 10$

The second statement also holds when ϕ is a random process, τ a stopping time, and ϵ and c random constants.

The amazing result here is that the time-change can transform hulls from simple curves to non-simple curves. In order to see the full affects of this theorem, we consider its affect on a few examples that have been time changed by the inverse of an α-stable subordinator. First, unlike in SLE_κ, when we time change Brownian motion, the resulting hull is no longer a simple curve for any κ. Second, the hull generated by $c\sqrt{t}$ is a ray that emanates from 0 and leaves at an angle dependent on c. When this is time-changed (see Figure 2), we have the following:

- if $\alpha < \frac{1}{2}$, the hull is not a simple curve
- if $\alpha > \frac{1}{2}$, the hull is a curve that leaves tangentially

Spacefilling Constant for the Loewner Equation

In the background section, a result from [LR12] is mentioned, which says if $\lambda \in$Lip(1/2) generates a spacefilling curve (i.e. has nonzero area), then $\|\lambda\|_{1/2} \geq 4.0001$. The value 4.0001 is not optimal. My goal is to find the optimal constant c so that if $\lambda \in$Lip(1/2) generates a spacefilling curve then $\|\lambda\|_{1/2} \geq c$.

Currently, I am working with a family of functions with parameter c with controllable Lip(1/2) norm, call them $\{\lambda_c\}_{c\geq 0}$. For each c, there is a sequence of recursively defined functions, $\{\xi_n^c\}_{n=1}^\infty$, that I have shown converge uniformly to λ_c. For any $c > 0$ and $n \in \mathbb{N}$, we show $\|\xi_n^c\|_{1/2} < 2c$. This result comes from carefully examining the sequence $\{\xi_n^c\}_{n=1}^\infty$. Then using the convergence, I show that $\|\lambda_c\|_{1/2} < 2c$ for $c > 0$.

I am examining the hulls that $\{\xi_n^c\}_{n=1}^\infty$ generate (see Figure 3 for an example) in order to investigate the hull generated by λ_c. One way I am doing this is by bounding ξ_n^c above and below by driving functions that control the growth of the hulls. This will guarantee that the trace cannot be spacefilling for particular values of c. For instance, if there is an interval $[a, b]$ contained in the closure of the hull that does not intersect the trace for each n, then the trace cannot be spacefilling. This technique has shown that for c in a certain range, the hull is not spacefilling. We will have a lower bound on c for this family to generate spacefilling curves. This work will give an upper bound on $\|\lambda\|_{1/2}$ that generate spacefilling curves.

Future Work

Questions related to the multiple Loewner equation:

- How strong is the convergence for the curves in the simulation method for the multiple Loewner equation? Is it possible that they converge uniformly (analogous to [Tra15])?
- If we start with multiple curves, can we simulate the driving functions and weights with methods similar to those in the chordal case by using the rapid, random oscillation?
- How do multiple hulls interact with each other's growth?
- If the driving functions collide, can we still get a constant weight and continuous driving functions that generate the hull? (This is still an open question for slits as well.)
- What do the constant weights tell us about the geometry of the hulls?

Questions related to time-changing the Loewner equation:

- When does the trace generated by a time-changed driving function exist?
- What are the probabilities that govern how the time-changed hulls interact with the real line or with themselves? For example, what is the probability that a time-changed hull never hits the real line?
- What is the geometry of the time-changed hull, for instance the Hausdorff dimension of the boundary?
- When the trace exists, is it reversible?
- What happens if we have a different type of time-change (e.g. discontinuous)?

Question related to spacefilling constant work: what is the optimal bound on $\|\cdot\|_{1/2}$ for driving functions that generate a spacefilling curve? The current work gives tools to analyze other families of curves that can be used to find the optimal constant. Further, the work identifies modifications that can be made in the current family which will provide a more optimal c value.

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