

## 4.8 Newton's Method

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**Example 4.8.1.** Given the following function in Figure 4.1, how can we approximate the  $x$ -value where  $f(x) = 0$ ?

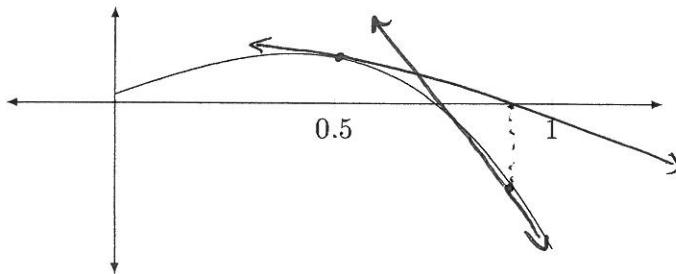


Figure 4.1: Graph of  $f(x)$

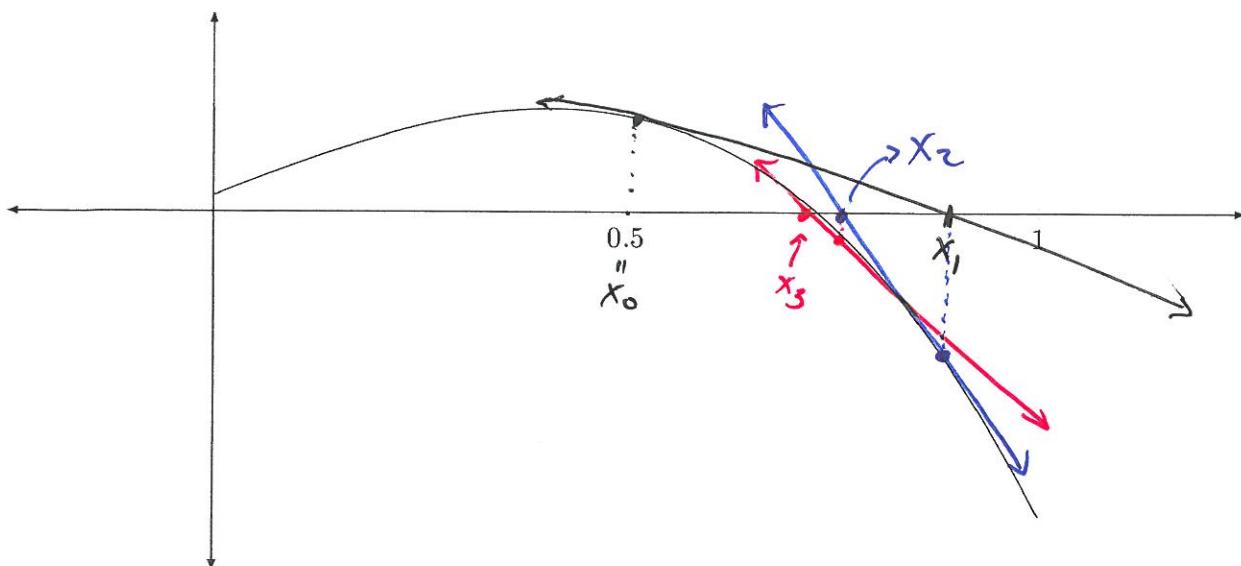
Old way: I V T

New Way: Newton's method

Step 1: initial guess, called  $x_0$

Step 2: use calculus to update guess

Step 3: repeat to desired accuracy

Figure 4.2: Graph of  $f(x)$ 

Formula:

Suppose  $x_{n-1}$

tangent line to  $f(x)$  at  $x_{n-1}$ :

$$y = f'(x_{n-1})(x - x_{n-1}) + f(x_{n-1})$$

want  $y=0$ :  $0 = f'(x_{n-1})(x - x_{n-1}) + f(x_{n-1})$

$$\Rightarrow -f(x_{n-1}) = f'(x_{n-1})(x - x_{n-1})$$

$$\Rightarrow -\frac{f(x_{n-1})}{f'(x_{n-1})} = (x - x_{n-1})$$

$$\Rightarrow \boxed{x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_n} \quad \text{Newton's method}$$

**Example 4.8.2.** For  $f(x) = \cos(2x) - \sin(x)$ , approximate the  $x$ -value in  $[0, \frac{\pi}{2}]$  where  $f(x) = 0$ .

$$x_0 = \frac{1}{2} = 0.5 \quad (\text{initial guess})$$

$$f'(x) = -2\sin(2x) - \cos x$$

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 0.5237751158 \dots \end{aligned}$$

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0.5235987846 \quad \text{accurate to 7 decimal places} \end{aligned}$$

$$x_3 = 0.5235987756$$

We can check our approximation by finding the exact solution.

*Hint:* Recall the trig identity  $\cos(2x) = 1 - 2\sin^2(x)$ .

$$\begin{aligned} f(x) &= \cos(2x) - \sin x \\ &= 1 - 2\sin^2 x - \sin x \\ &= -2\sin^2 x - \sin x + 1 \\ u = \sin x &\quad = -2u^2 - u + 1 \end{aligned}$$

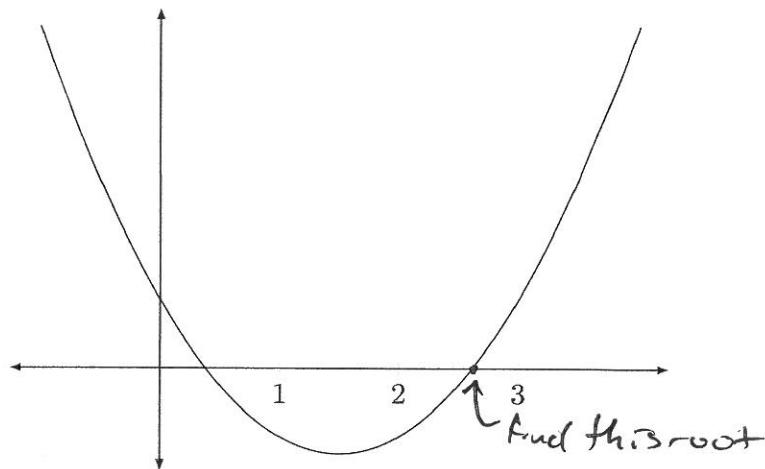
$$\sin x = u = \frac{1 \pm \sqrt{1 - 4 \cdot (-2) \cdot 1}}{2(-2)} = \frac{-1 \pm 3}{4}$$

$$\begin{array}{ll} \sin x = \frac{+2}{4} \text{ or } \sin x = \frac{-4}{4} = -1 \\ \text{or} \quad \frac{+1}{2} \quad \quad \quad \uparrow \text{not on } [0, \pi/2] \end{array}$$

$$\sin(\pi/6) = \frac{1}{2}$$

$$\Rightarrow \text{root } x = \frac{\pi}{6} = 0.5235987756 \dots$$

**Example 4.8.3.** For  $g(x) = x^2 - 3x + 1$ , use Newton's method to approximate the largest  $x$  satisfying  $g(x) = 0$ .

Figure 4.3: Graph of  $g(x)$ 

$$x_0 = 3$$

$$g'(x) = 2x - 3$$

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = 2.66$$

$$x_2 = 2.619047619$$

$$x_3 = 2.618034448$$

$$x_4 = 2.618033989$$

actual sol:  $x = \frac{3 + \sqrt{5}}{2} = 2.618033989\dots$

**Example 4.8.4.** In the previous example, what points would be a “bad” choice for  $x_0$ ?

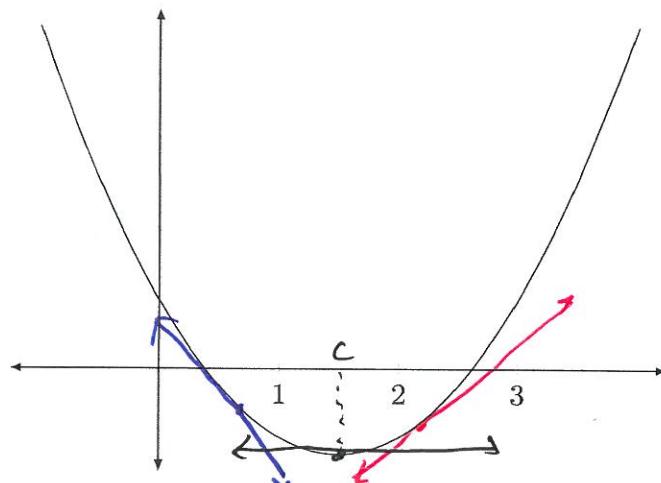


Figure 4.4: Graph of  $g(x)$

$$\cancel{g'(c) = 0}$$
$$\cancel{x - \frac{g(c)}{g'(c)}}$$