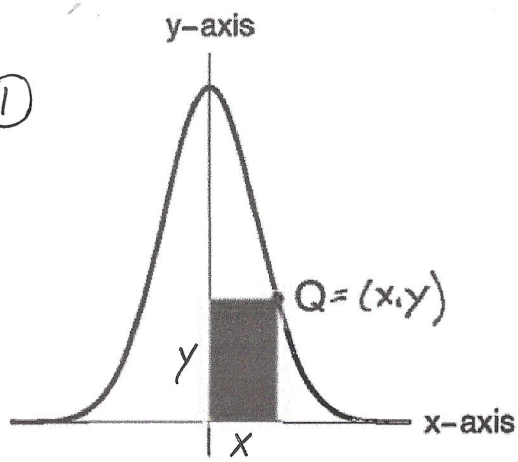
Example 4.7.4. Construct a rectangle in the first quadrant with one corner at the origin and the opposite corner on the graph of the function $y = 10e^{-x^2}$ as illustrated. Find the coordinates of the point Q that give the rectangle the largest area.

(1)



(2) $A = \text{area of rectangle} = xy$

(3) $y = 10e^{-x^2}$

$\Rightarrow A = 10xe^{-x^2}$

(4) $x \in (0, \infty)$

(5) $A' = 10e^{-x^2} - 20x^2e^{-x^2} = 10e^{-x^2}(1-2x^2)$

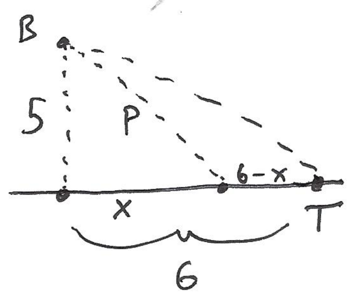
$A' = 0 \Rightarrow 1 - 2x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}} \Rightarrow CP: x = \frac{1}{\sqrt{2}}$

(6) $\underset{0}{(}\overset{+}{\longrightarrow}\underset{\frac{1}{\sqrt{2}}}{|}\overset{-}{\longrightarrow} A'$ $\Rightarrow$ max occurs at $x = \frac{1}{\sqrt{2}}$

$\Rightarrow Q = \left(\frac{1}{\sqrt{2}}, 10e^{-\left(\frac{1}{\sqrt{2}}\right)^2}\right) = \left(\frac{1}{\sqrt{2}}, 10e^{-1/2}\right)$

= coordinates of point that give rectangle of largest area

Example 4.7.5. Brooke is located 5 miles out at sea from a straight shoreline in her kayak. She wants to make it to the taco truck on the beach for lunch, which is 6 miles from the point on the shore closest to Brooke. Brooke can paddle 2 miles/hour and walk 4 miles/hour. Let's assume that Brooke will paddle along a straight line to the shore. Where should she land her boat to eat as soon as possible?

① 

x: dist. from pt on shore closest to Brooke to the pt where she lands

6-x: dist Brooke walks

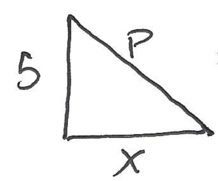
p: dist. Brooke paddles

a: time Brooke paddles

b: time Brooke walks

② T: total time = a+b

③ $a = \frac{\text{dist paddled}}{\text{rate}} = \frac{p}{2} = \frac{\sqrt{25+x^2}}{2}$



$\Rightarrow 5^2 + x^2 = p^2$

$\Rightarrow p = \sqrt{25+x^2}$

$b = \frac{\text{dist walked}}{\text{rate}} = \frac{6-x}{4}$

$\Rightarrow T(x) = \frac{\sqrt{25+x^2}}{2} + \frac{6-x}{4}$

④ $x \in [0, 6]$

⑤ $T'(x) = \frac{x}{2\sqrt{25+x^2}} - \frac{1}{4}$

$T'(x) = 0 \Rightarrow x = \pm \frac{5}{\sqrt{3}} \Rightarrow CP: x = \frac{5}{\sqrt{3}}$

⑥ $T(0) = 4$

$T(5/\sqrt{3}) = 3.665 \leftarrow \min$

$T(6) = 3.905$

$\Rightarrow$ Brooke should land $5/\sqrt{3}$ miles from the point on shore closest to her.

Example 4.7.6. A car rental agency rents 180 cars per day at a rate of \$32 per day. For each \$1 increase in rate, 5 fewer cars are rented. At what rate should the cars be rented to produce the maximum income?

(1) x = dollar increase in rate

n = # of cars rented per day

(2) I = income = (rate)$\cdot$(# cars rented) = $(x+32)n$

(3) $n = 180 - 5x$ (5 fewer cars rented with each increase in x)

$\Rightarrow I = (x+32)(180-5x)$

(4) need $x \geq 0$ and $180 - 5x \geq 0 \Rightarrow x \in [0, 36]$

(5) $I' = 180 - 5x - 5(x+32)$

$= 20 - 10x$

$I' = 0 \Rightarrow x = 2$ (CP)

(6) $I(0) = 32 \cdot 180 = \$5760$/day

$I(2) = 34 \cdot 170 = \$5780$/day $\leftarrow$ max

$I(36) = \$0$/day

$\Rightarrow$ Cars should be rented at \$34/day to maximize profit.