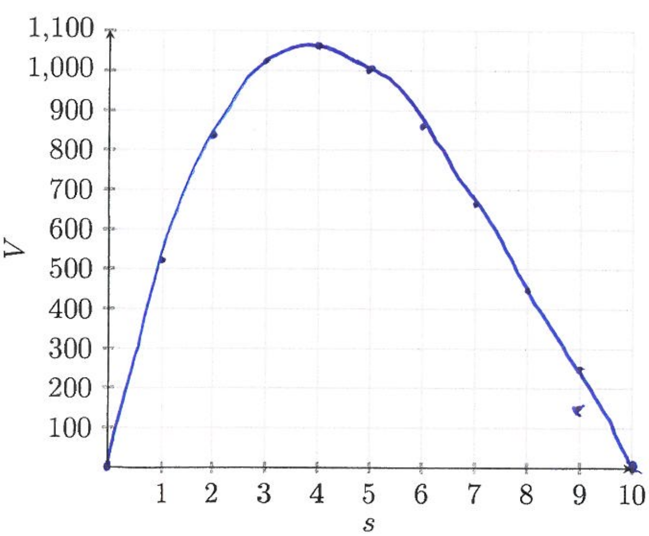
4.7 Applied Optimization

Example 4.7.1. Your group is given a piece of graph paper that was originally 20 units by 30 units. Four congruent squares have been cut from each corner. The size of the squares is different for each group. Fold the sides of your paper up to make a box with an open top.

Record the following (in graph paper units).

	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$	$s=9$
Length of box:	28	26	24	22	20	18	16	14	12
Width of box:	18	16	14	12	10	8	6	4	2
Height of box:	1	2	3	4	5	6	7	8	9
Volume of box:	504	832	1008	1056	1000	864	672	448	216

Now let's plot a graph of the volumes, V, each group found based on the length of the side of the square, s, cut out from each corner.



Based on this graph, what is the approximate maximum volume?

≈ 1056 units^3

Approximately what size square cut out of the corners results in this maximum volume?

≈ 4 units

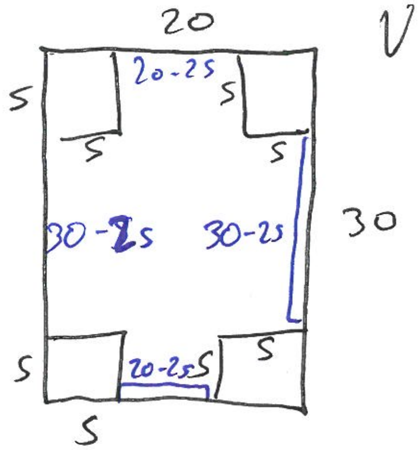
How could you use calculus to find the exact maximum volume?

If we had a function for Volume, we could take derivative and find CP and then determine where the maximum occurs.

Observe that the volume is determined by the side length s of the square cut from each corner (a quantity that we can control). We will now use calculus to figure out exactly what size square to cut from each corner in order to maximize the volume of the open-topped box.

Finding the maximum volume using calculus:

(a) Write a function for the volume (that will depend on s).



$V = lwh$

$l = 30 - 2s$

$w = 20 - 2s$

$h = s$

$V = (30-2s)(20-2s)s$

$\Rightarrow V(s) = 600s - 100s^2 + 4s^3$

(b) What is the domain of your function that fits the context of this problem?

$0 \leq s \leq 10$ $\qquad [0, 10]$

(c) Find the maximum volume using calculus.

$V'(s) = 12s^2 - 200s + 600$

$$s = \frac{200 \pm \sqrt{200^2 - 4 \cdot 600 \cdot 12}}{2 \cdot 12}$$

$\Rightarrow s \approx 12.7429 \rightarrow$ not in domain

$s \approx 3.92375$

$V(0) = 0$

$V(3.92375) = 1056.31$

$V(10) = 0$

max Volume: 1056.31 units^3

The methods you have learned so far for finding extreme values have practical applications in many areas of life:

- A businessperson wants to minimize costs and maximize profits.
- A traveler wants to minimize transportation time.
- A migrating bird wants to maximize the distance it can travel without stopping, given the energy that can be stored as body fat.

In this section, we will apply your knowledge of calculating absolute minimums and absolute maximums to solve **optimization problems.**

Goal of an optimization problem: Make a decision that is best possible for some quantity (maximize/minimize that quantity)

The function that we are trying to maximize or minimize is called the objective function. Oftentimes, but not in every situation, our goal is restricted by something that we refer to as a **constraint**. Examples:

- A businessperson wants to maximize his or her profits but is constrained by
  market, money, profit, time, etc.
- A traveler wants to minimize transportation time but is constrained by
  traffic, speed limit, etc.

Steps for Solving Optimization Problems:

picture **1.** Draw a picture and choose your variables.

dependent variable: the one(s) you're maximizing or minimizing

independent variable(s): the one(s) you control

eqn **2.** Create your objective function by relating your variables.

solve for 1 variable **3.** If the **objective function** is written as a function of **more than one variable**, then you will need to use a **constraint equation**. Solve your constraint equation for one variable and plug that in to your objective function so that it becomes a function of only one variable.

domain **4.** Find the domain of your objective function that fits the context of the problem.

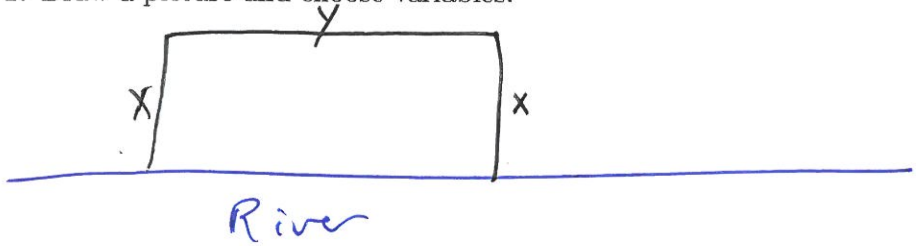
Use calculus tools to find an extreme value of the function on the domain:

CP **5.** Find derivative and critical points (Do **not** stop at this step! You must verify whether or not the critical point gives a global max or min.)

max/min **6.** Find global min or max

Example 4.7.2. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

1. Draw a picture and choose variables.



2. Create your objective function.

$$A = xy$$

3. Get your objective function into one variable using a constraint equation.

$$2x + y = 2400 \text{ (const. eqn)}$$

$$\Rightarrow y = 2400 - 2x$$

$$\Rightarrow A = xy = (x)(2400 - 2x) = 2400x - 2x^2$$

4. Find the domain of your objective function.

$0 \le y$

$0 \le x \le 1200$ $\Rightarrow$ $y = 2400 - 2x \ge 0 \Rightarrow 2400 \ge 2x$

$\Rightarrow 1200 \ge x$

Use calculus to find the maximum area.

$$A'(x) = 2400 - 4x$$

$$A'(x) = 0 \Leftrightarrow x = 600$$

$A(0) = 0$

$A(600) = 720{,}000 \text{ ft}^2$ $\Rightarrow$

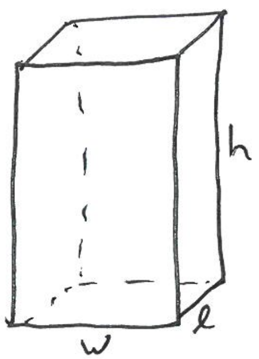
$A(1200) = 0$

Dimensions of field w/ largest area:

$x = 600$ ft, $y = 2400 - 2 \cdot 600 = 1200$ ft

Example 4.7.3. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

1



2 $\left.\begin{array}{l} 10 = V = lwh \\ l = 2w \end{array}\right\}$ constraint eqns

$C = 10\cdot wl + 6\cdot lh\cdot 2 + 6\cdot wh\cdot 2$

obj (want to minimize)

3 $l = 2w \Rightarrow 10 = (2w)wh = 10 = 2w^2h$

$\Rightarrow h = \frac{10}{2w^2} = \frac{5}{w^2}$

$\Rightarrow C(w) = 10w(2w) + 12(2w)\left(\frac{5}{w^2}\right) + 12w\left(\frac{5}{w^2}\right)$

$= 20w^2 + \frac{120}{w} + \frac{60}{w}$

$= 20w^2 + \frac{180}{w} = 20w^2 + 180w^{-1}$

4 $0 < l, w, h$ ~~AND~~ $< \infty$

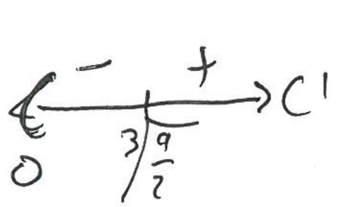
~~$10 > l = 2w \Rightarrow w < 5$~~

5 $C'(w) = 40w - 180w^{-2}$

$40w - 180w^{-2} = 0 \Rightarrow 40w = 180w^{-2}$

$\Rightarrow w^3 = \frac{180}{40}$

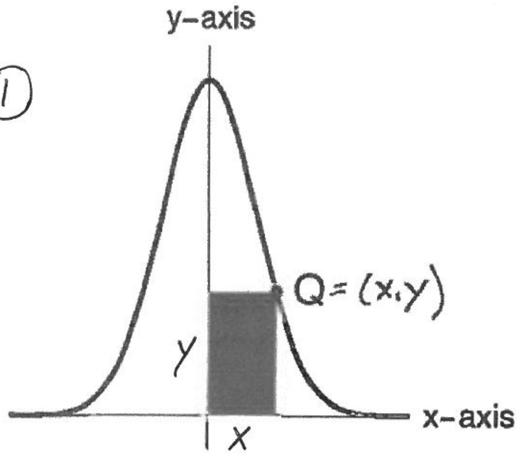
$\Rightarrow w = \sqrt[3]{\frac{180}{40}} = \sqrt[3]{\frac{9}{2}}$



$\boxed{C\left(\sqrt[3]{\frac{9}{2}}\right) = \$163.54 \text{ minimum cost}}$

Example 4.7.4. Construct a rectangle in the first quadrant with one corner at the origin and the opposite corner on the graph of the function $y = 10e^{-x^2}$ as illustrated. Find the coordinates of the point Q that give the rectangle the largest area.

(1)



(2) A = area of rectangle = xy

(3) $y = 10e^{-x^2}$

$\Rightarrow A = 10xe^{-x^2}$

(4) $x \in (0, \infty)$

(5) $A' = 10e^{-x^2} - 20x^2e^{-x^2} = 10e^{-x^2}(1-2x^2)$

$A' = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}} \Rightarrow CP: x = \frac{1}{\sqrt{2}}$

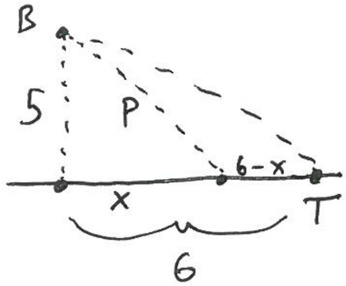
(6) $\underset{0}{(}\overset{+}{\text{———}}\underset{\frac{1}{\sqrt{2}}}{|}\overset{-}{\text{———}}\rightarrow A'$ $\Rightarrow$ max occurs at $x = \frac{1}{\sqrt{2}}$

$\Rightarrow Q = \left(\frac{1}{\sqrt{2}}, 10e^{-\left(\frac{1}{\sqrt{2}}\right)^2}\right) = \left(\frac{1}{\sqrt{2}}, 10e^{-1/2}\right)$

= coordinates of point that give rectangle of largest area

Example 4.7.5. Brooke is located 5 miles out at sea from a straight shoreline in her kayak. She wants to make it to the taco truck on the beach for lunch, which is 6 miles from the point on the shore closest to Brooke. Brooke can paddle 2 miles/hour and walk 4 miles/hour. Let's assume that Brooke will paddle along a straight line to the shore. Where should she land her boat to eat as soon as possible?

(1)



x: dist. from pt on shore closest to Brooke to the pt where she lands

$6-x$: dist Brooke walks

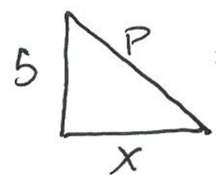
p: dist. Brooke paddles

a: time Brooke paddles

b: time Brooke walks

(2) T: total time $= a+b$

(3) $a = \dfrac{\text{dist paddled}}{\text{rate}} = \dfrac{p}{2} = \dfrac{\sqrt{25+x^2}}{2}$



$\Rightarrow 5^2 + x^2 = p^2$

$\Rightarrow p = \sqrt{25+x^2}$

$b = \dfrac{\text{dist walked}}{\text{rate}} = \dfrac{6-x}{4}$

$\Rightarrow T(x) = \dfrac{\sqrt{25+x^2}}{2} + \dfrac{6-x}{4}$

(4) $x \in [0, 6]$

(5) $T'(x) = \dfrac{x}{2\sqrt{25+x^2}} - \dfrac{1}{4}$

$T'(x) = 0 \Rightarrow x = \pm \dfrac{5}{\sqrt{3}} \Rightarrow CP: x = \dfrac{5}{\sqrt{3}}$

(6) $T(0) = 4$

$T(5/\sqrt{3}) = 3.665 \leftarrow \text{min}$

$T(6) = 3.905$

$\Rightarrow$ Brooke should land $5/\sqrt{3}$ miles from the point on shore closest to her.

Example 4.7.6. A car rental agency rents 180 cars per day at a rate of \$32 per day. For each \$1 increase in rate, 5 fewer cars are rented. At what rate should the cars be rented to produce the maximum income?

1. x = dollar increase in rate

   n = # of cars rented per day

2. I = income = (rate)$\cdot$(# cars rented) = $(x+32)n$

3. $n = 180 - 5x$ (5 fewer cars rented with each increase in x)

   $\Rightarrow I = (x+32)(180-5x)$

4. need $x \geq 0$ and $180 - 5x \geq 0 \Rightarrow x \in [0, 36]$

5. $I' = 180 - 5x - 5(x+32)$

   $= 20 - 10x$

   $I' = 0 \Rightarrow x = 2$ (CP)

6. $I(0) = 32 \cdot 180 = \$5760$/day

   $I(2) = 34 \cdot 170 = \$5780$/day ← max

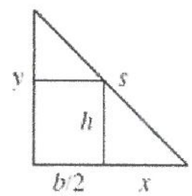
   $I(36) = \$0$/day

   $\Rightarrow$ cars should be rented at \$34/day to maximize profit.

Section 4.7 Homework Hints

4.7 HW 1 #4: The wire is divided into 2 pieces. Let one piece (that is used to form the circle) be of length x and the other piece (that is used to form the square) be of length $60 - x$, where 60 is the total length of the wire (different for each student). Then, x is equal to the circumference of a circle and $60 - x$ is equal to the perimeter of a square. You can use these relationships to solve for r and s in terms of x to then plug in to your objective function.

4.7 HW 3 #4: Consider the right triangle formed by the right half of the rectangle and its roof. This triangle has hypotenuse s.



As shown, let y be the height of the roof and let x be the distance from the right base of the rectangle to the base of the roof. Use similar triangles to solve for y in terms of x (and go ahead and plug in your given values for h and b so that your y is only in terms of x.)

Then, y, x, and s are related by the Pythagorean Theorem. This allows you to write an equation for s^2 in terms of only x (since you can write y in terms of x.)

Key Idea: Since $s > 0$, s^2 is least whenever s is least, so you can minimize s^2 instead of s. (This means you can take the derivative of your expression for s^2 and set it equal to 0 and solve for its critical points to find the minimum). Once you do this, you will have the smallest s^2, and you will just need to take the square root of that to find the smallest s.