

## 4.6 Curve Sketching

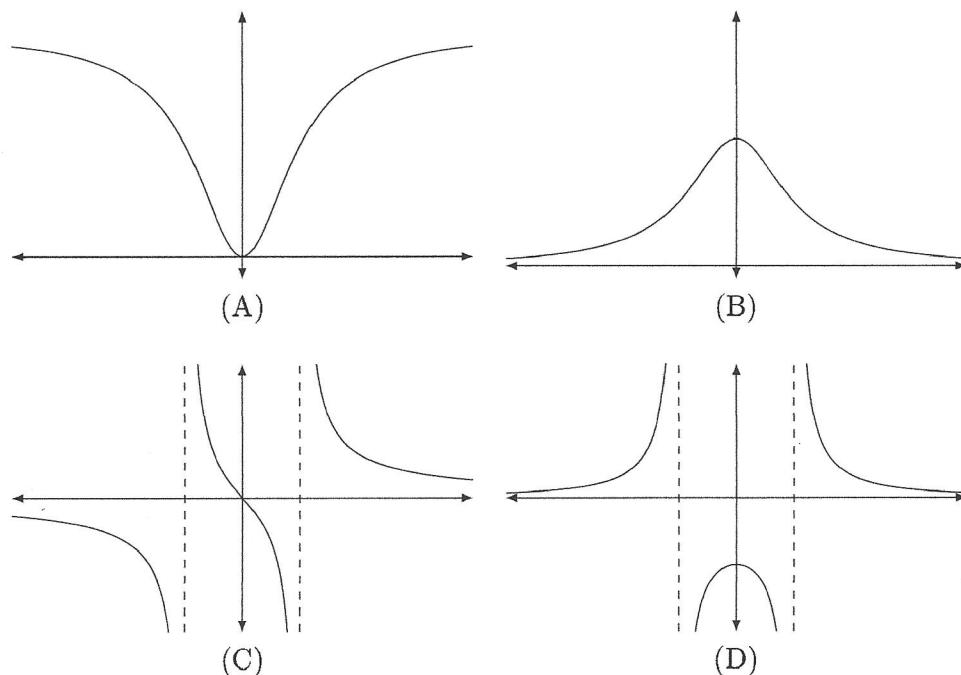
**Example 4.6.1.** Match the functions with their graphs.

(a)  $y = \frac{1}{x^2 - 1}$   $\rightarrow$  VA:  $x = \pm 1$ ,  $y(0) = -1$   $\leftrightarrow$  (D)

(b)  $y = \frac{x^2}{x^2 + 1}$   $\rightarrow$  no VA,  $y(0) = 0$   $\leftrightarrow$  (A)

(c)  $y = \frac{1}{x^2 + 1}$   $\rightarrow$  no VA,  $y(0) = 1$   $\leftrightarrow$  (B)

(d)  $y = \frac{x}{x^2 - 1}$   $\rightarrow$  VA:  $x = \pm 1$ ,  $y(0) = 0$   $\leftrightarrow$  (C)



Recall:



$$f' < 0$$

$$f'' > 0$$



$$f' > 0$$

$$f'' > 0$$



$$f' > 0$$

$$f'' < 0$$



$$f' < 0$$

$$f'' < 0$$

**Example 4.6.2.** Find transition points, intervals of increasing/decreasing, concavity, and asymptotic behavior of  $f(x) = 4x^3 - 12x^2$ , then sketch the graph with this information.

$$f(x) = 4x^3 - 12x^2 \quad \text{domain: } (-\infty, \infty), \text{ zeros: } \boxed{x=0, 3}$$

$$= 4x^2(x-3) \quad \begin{array}{c} \leftarrow \quad \leftarrow \quad + \\ \text{---} \quad \text{---} \quad \text{---} \\ 0 \quad \quad \quad 3 \end{array} \rightarrow f$$

$$f'(x) = 12x^2 - 24x = 12x(x-2)$$

$$\text{CP: } x=0, 2 \quad \begin{array}{c} + \quad - \quad + \\ \text{---} \quad \text{---} \quad \text{---} \\ 0 \quad \quad \quad 2 \end{array} \rightarrow f'$$

$$f_{\text{inc}}: (-\infty, 0) \cup (2, \infty) \quad \text{local min: } \boxed{(2, -16)}$$

$$f_{\text{dec}}: (0, 2) \quad \text{local max: } \boxed{(0, 0)}$$

$$f''(x) = 24x - 24 = 24(x-1)$$

$$f''(x) = 0 \Leftrightarrow x=1$$

$$\begin{array}{c} - \quad + \\ \text{---} \quad \text{---} \\ 1 \end{array} \rightarrow f''$$

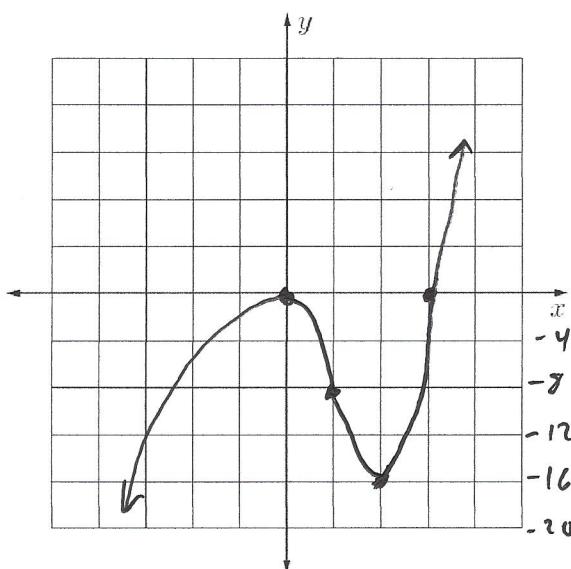
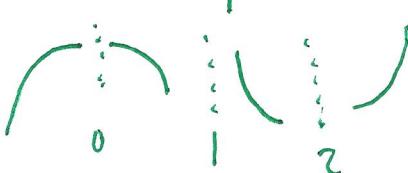
$$f_{\text{conc. up: }} (1, \infty)$$

$$f_{\text{conc. down: }} (-\infty, 1)$$

$$\text{IP: } \boxed{(1, -8)}$$

$$\begin{array}{c} + \quad - \quad + \\ \text{---} \quad \text{---} \quad \text{---} \\ 0 \quad \quad \quad 2 \end{array} \rightarrow f'$$

$$\begin{array}{c} - \quad + \\ \text{---} \quad \text{---} \\ 1 \end{array} \rightarrow f''$$



**Example 4.6.3.** Find transition points, intervals of increasing/decreasing, concavity, and asymptotic behavior of  $g(x) = \frac{4x}{x^2 + 1}$ , then sketch the graph with this information.

$$g(x) = \frac{4x}{x^2 + 1}$$

$\leftarrow + \rightarrow g$

0

Domain:  $(-\infty, \infty)$ , HA:  $y=0$ , zeros:  $x=0$

$$\lim_{x \rightarrow \infty} \frac{4x}{x^2 + 1} \stackrel{\text{Hop}}{=} \lim_{x \rightarrow \infty} \frac{4}{2x} = 0$$

$$g'(x) = \frac{4(x^2 + 1) - (2x)(4x)}{(x^2 + 1)^2} = \frac{-4x^2 + 4}{(x^2 + 1)^2} = \frac{-4(x^2 - 1)}{(x^2 + 1)^2} = \frac{-4(x+1)(x-1)}{(x^2 + 1)^2}$$

$$(P: x = \pm 1 \quad \leftarrow + \rightarrow g')$$

-1 1

$$g \text{ mc: } (-1, 1)$$

$$\text{local max: } (1, 2)$$

$$g \text{ dec: } (-\infty, -1) \cup (1, \infty)$$

$$\text{local min: } (-1, -2)$$

$$g''(x) = \frac{(-8x)(x^2 + 1)^2 - 2(x^2 + 1)(2x)(-4x^2 + 4)}{(x^2 + 1)^4}$$

$$= \frac{(-8x)(x^2 + 1) - 4x(-4x^2 + 4)}{(x^2 + 1)^3} = \frac{-8x^3 - 8x + 16x^3 - 16x}{(x^2 + 1)^3}$$

$$= \frac{8x^3 - 24x}{(x^2 + 1)^3} = \frac{8x(x^2 - 3)}{(x^2 + 1)^3}$$

$$g''(x) = 0 \Leftrightarrow x = 0, \pm\sqrt{3}$$

$\leftarrow - + - + \rightarrow g''$

- $\sqrt{3}$  0  $\sqrt{3}$

$\leftarrow + - + + \rightarrow 8x$

- $\sqrt{3}$  0  $\sqrt{3}$

$$g \text{ conc up: } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

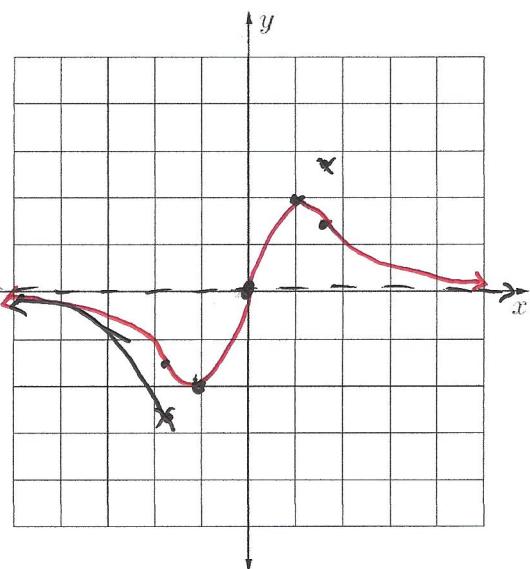
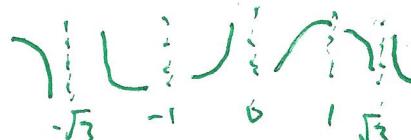
$$g \text{ conc down: } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

$$\text{IP: } [(-\sqrt{3}, -\sqrt{3}), (0, 0), (\sqrt{3}, \sqrt{3})]$$

$\leftarrow + \rightarrow g'$

$\leftarrow + - + \rightarrow g''$

- $\sqrt{3}$  0  $\sqrt{3}$



$x(2x-3)$ 

**Example 4.6.4.** Find transition points, intervals of increasing/decreasing, concavity, and asymptotic behavior of  $h(x) = \frac{2x^2 - 3x}{x-2}$ , then sketch the graph with this information.

$$h = \frac{2x^2 - 3x}{x-2}, \text{ Domain of } h: x \neq 2, \text{ VA at } x=2, x=0, \frac{3}{2}$$

$$h' = \frac{(4x-3)(x-2) - (2x^2 - 3x)}{(x-2)^2} = \frac{2(x-1)(x-3)}{(x-2)^2} \quad \left( = \frac{2x^2 - 8x + 6}{(x-2)^2} \right)$$

$$h'' = \frac{(4x-8)(x-2)^2 - 2(x-2)(2x^2 - 8x + 6)}{(x-2)^4} = \frac{4}{(x-2)^3}$$

CP:  $\underline{\text{not CP}}$   $(2, 1, 3) = x$   $\begin{array}{c} + \\[-1ex] | \\[-1ex] 2 \\[-1ex] | \\[-1ex] 3 \end{array} \rightarrow h'$

$h_{\text{inc}}: (-\infty, 1) \cup (3, \infty) \quad \text{loc max: } (1, 1)$

$h_{\text{dec}}: (1, 2) \cup (2, 3) \quad \text{loc min: } (3, 9)$

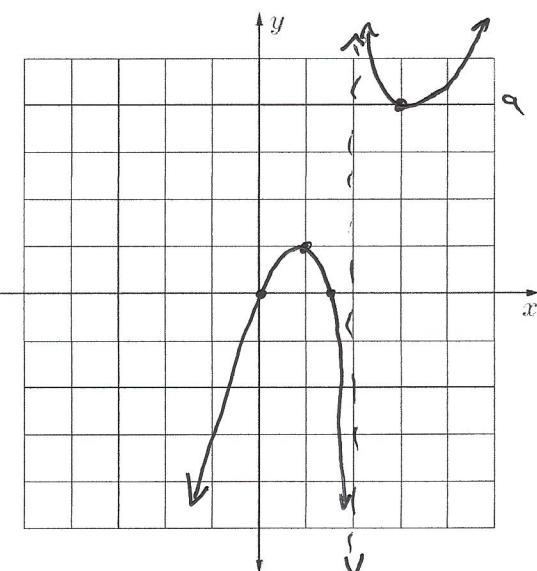
$h'' = 0 \text{ never zero, } h'' \text{ undefined } x=2$

$\begin{array}{c} + \\[-1ex] | \\[-1ex] 2 \\[-1ex] | \\[-1ex] \rightarrow h'' \end{array}$

no IP

$h_{\text{conc up:}} (2, \infty)$

$h_{\text{conc down:}} (-\infty, 2)$



**Example 4.6.5.** Find transition points, intervals of increasing/decreasing, concavity, and asymptotic behavior of  $y = x - 3x^{1/3}$ , then sketch the graph with this information.

$$y = x - 3x^{1/3} \Rightarrow \text{domain: } (-\infty, \infty), \text{ zeros: } x = 0, \pm 3\sqrt[3]{3}, \text{ no asymptotes}$$



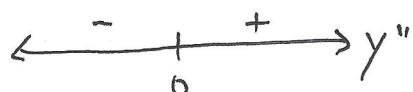
$$y' = 1 - x^{-2/3} \rightarrow [0 = 1 - x^{-2/3} \Rightarrow x = \pm 1, \text{ undefined at } x = 0]$$

$$\Rightarrow \text{CP: } x = 0, \pm 1 \quad \leftarrow + \begin{matrix} - \\ 0 \end{matrix} - \begin{matrix} + \end{matrix} \rightarrow y'$$

$$y \text{ inc: } (-\infty, -1) \cup (1, \infty) \quad \text{local min: } (-1, 2)$$

$$y \text{ dec: } (-1, 0) \cup (0, 1) \quad \text{local max: } (1, 2)$$

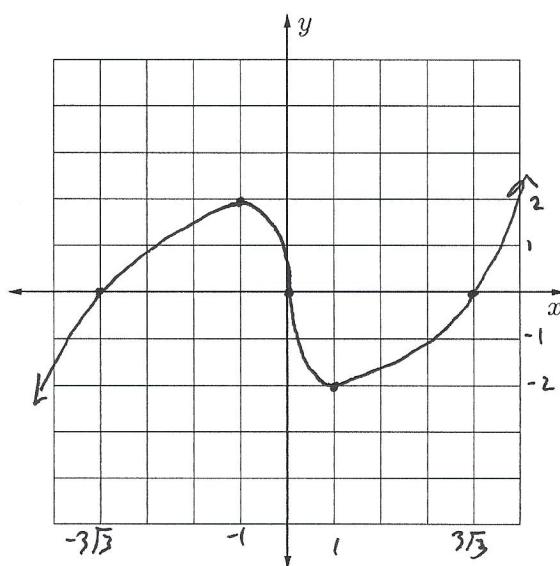
$$y'' = \frac{2}{3}x^{-5/3}, \quad y'' \text{ never equals 0 and is undefined at } x = 0$$



$$y \text{ concave up: } (0, \infty)$$

$$y \text{ concave down: } (-\infty, 0)$$

$$\text{IP: } (0, 0)$$



# How to do curve sketching

① Take 1st, 2nd derivatives

② find domain of  $f$

③ HA/VA of  $f$

$f$

④ zeros of  $f$

⑤ Sign chart for  $f$

⑥ find CP (zeros or undefined pts of  $f'$ )

⑦ sign chart for  $f'$

$f'$

⑧ inc/dec, loc min/max

⑨ zeros or undefined pts of  $f''$  (possible IP)

⑩ sign chart for  $f''$

$f''$

⑪ conc up/down IP

⑫ Sketched

If  $f(x)$  is concave down, then  $f'(x)$  is (circle all that apply):

increasing       decreasing       constant

Is there anything else that can be said about  $f'(x)$ ?

$$f \text{ conc down} \Leftrightarrow f'' < 0$$

$$\begin{matrix} / \\ (f')' \end{matrix}$$

$\Leftrightarrow f'$  decreasing

