

4.4 The Shape of a Graph

Definition 4.4.1. We say $f(x)$ is concave up at $x = c$ if $f''(c) > 0$.

We say $f(x)$ is concave down at $x = c$ if $f''(c) < 0$.

We say $(c, f(c))$ is an inflection point if $f(x)$ changes concavity at $x = c$.

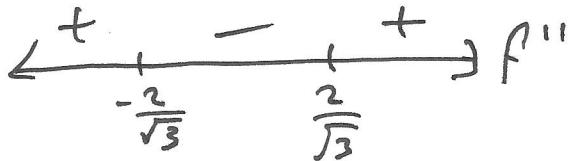
Example 4.4.1. Let $f(x) = \frac{1}{4}x^4 - 2x^2$. Where is $f(x)$ concave up or down and what are the inflection points of $f(x)$?

$$f'(x) = x^3 - 4x$$

$$f''(x) = 3x^2 - 4$$

$$3x^2 - 4 = 0$$

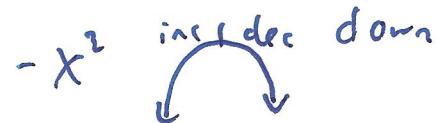
$$x^2 = \frac{4}{3} \Rightarrow x = \pm \sqrt{\frac{4}{3}}$$



$$\text{Concave up: } (-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$$

$$\text{Concave down: } (-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$$

$$\text{IP: } \left(-\frac{2}{\sqrt{3}}, f\left(-\frac{2}{\sqrt{3}}\right)\right), \left(\frac{2}{\sqrt{3}}, f\left(\frac{2}{\sqrt{3}}\right)\right)$$



$$\begin{cases} f' > 0 \\ f'' > 0 \end{cases}$$

$$\begin{cases} f' < 0 \\ f'' \geq 0 \end{cases}$$

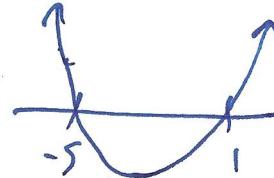
$$\begin{cases} f' > 0 \\ f'' < 0 \end{cases}$$

$$\begin{cases} f' < 0 \\ f'' < 0 \end{cases}$$

Example 4.4.2. Let $y = (x^2 - 7)e^x$. Find intervals of concavity and inflection points.

$$\begin{aligned}y' &= 2xe^x + (x^2 - 7)e^x \\&= (x^2 + 2x - 7)e^x\end{aligned}$$

$$\begin{aligned}y'' &= (2x+2)e^x + (x^2 + 2x - 7)e^x \\&= (x^2 + 4x - 5)e^x\end{aligned}$$



$$0 = x^2 + 4x - 5 = (x+5)(x-1)$$

$$\Rightarrow x = -5, 1$$

$$\begin{array}{c} + \quad - \quad + \\ \leftarrow \qquad \qquad \rightarrow \\ -5 \qquad \qquad 1 \end{array} f''$$

Concave up: $(-\infty, -5) \cup (1, \infty)$

Concave down: $(-5, 1)$

$$\text{IP: } (-5, y(-5)) = (-5, 18e^{-5})$$

$$(1, y(1)) = (1, -6e)$$

Recall: First Derivative Test:

$$\leftarrow \underset{c}{\underset{\text{---}}{|}} \rightarrow f' \Rightarrow (c, f(c)) \text{ is a local max}$$

$$\leftarrow \underset{c}{\underset{\text{---}}{|}} \rightarrow f' \Rightarrow (c, f(c)) \text{ is a local min}$$

Theorem 4.4.1 (Second Derivative Test). If $x = c$ is a critical point of $f(x)$, then:

$$f''(c) < 0 \Rightarrow (c, f(c)) \text{ is a local max}$$

$$f''(c) > 0 \Rightarrow (c, f(c)) \text{ is a local min}$$

$$f''(c) = 0 \Rightarrow \text{nothing}$$

~~W.M.S.~~

Example 4.4.3. Let $f(x) = \frac{1}{4}x^4 - 2x^2$. Find critical points and use the second derivative test to classify them.

$$f'(x) = x^3 - 4x$$

$$0 = x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$$

$$CP: x = 0, 2, -2$$

$$f''(x) = 3x^2 - 4$$

$$f''(-2) = 8 > 0$$

$$f''(0) = -4 < 0$$

$$f''(2) = 8 > 0$$

by 2nd der. test

min at $x = -2$

max at $x = 0$

min at $x = 2$