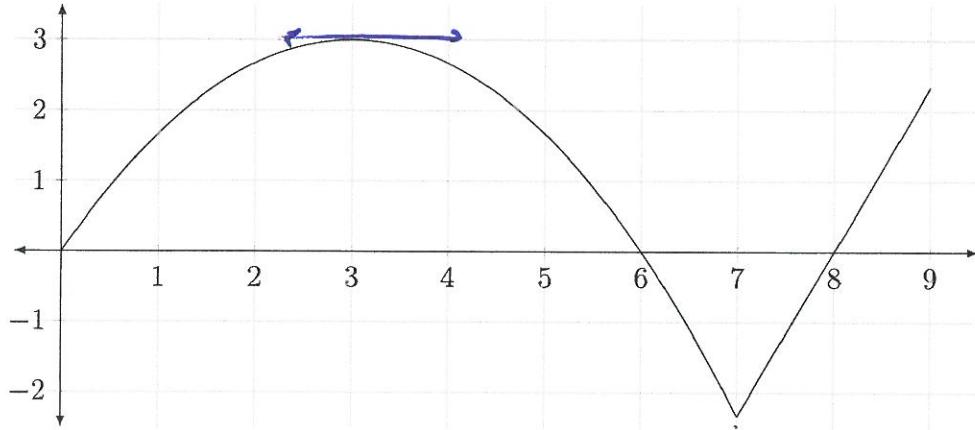


4.2 Extreme Values

Example 4.2.1. The graph of $f(x)$ is given below.



a. What is the maximum of $f(x)$?

$$3$$

b. What is the minimum of $f(x)$?

$$-2.5$$

c. What happens at these points?

$$f'(x) = 0 \text{ or undefined}$$

Definition 4.2.1 (Critical Point). We say $x = c$ is a critical point of $f(x)$ when

$$f'(c) = 0 \text{ or } f'(c) \text{ is undefined}$$

Definition 4.2.2 (Extreme Values). The extreme values of $f(x)$ on $[a, b]$ are the minimum and maximum values of $f(x)$ for $x \in [a, b]$.

y values

Definition 4.2.3. $f(c)$ is a local minimum means for input values x that are “close” to c , we have

$$f(x) \geq f(c)$$

Definition 4.2.4. $f(c)$ is a local maximum means for input values x that are “close” to c , we have

$$f(x) \leq f(c)$$

$\rightarrow f' = 0 \text{ or undefined.}$

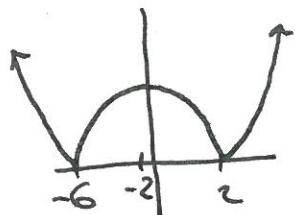
Example 4.2.2. Find the critical points of $f(x) = xe^{2x}$.

$$f'(x) = e^{2x} + 2xe^{2x}$$

$$0 = e^{2x} + 2xe^{2x} = e^{2x}(1+2x)$$

$$\Rightarrow x = -\frac{1}{2} \text{ is only CP}$$

Example 4.2.3. Find the critical points of $g(x) = |x^2 + 4x - 12| = |(x+6)(x-2)|$



$f'(x)$ undefined at $x = -6, 2$

for $-6 < x < 2$, $f(x) = -(x^2 + 4x - 12)$, so

$$f'(x) = -2x - 4$$

$$-6 < x < 2$$

$$-2x - 4 = 0$$

$$-(x^2 + 4x - 12)$$

$$\Rightarrow x = -2$$

$$CP: x = -6, -2, 2$$

Example 4.2.4. Find the critical points of $h(x) = x + 6x^{\frac{1}{3}}$.

$$h'(x) = 1 + 2x^{-\frac{2}{3}}$$

$h'(x)$ undefined at $x = 0$

$$CP: x = 0$$

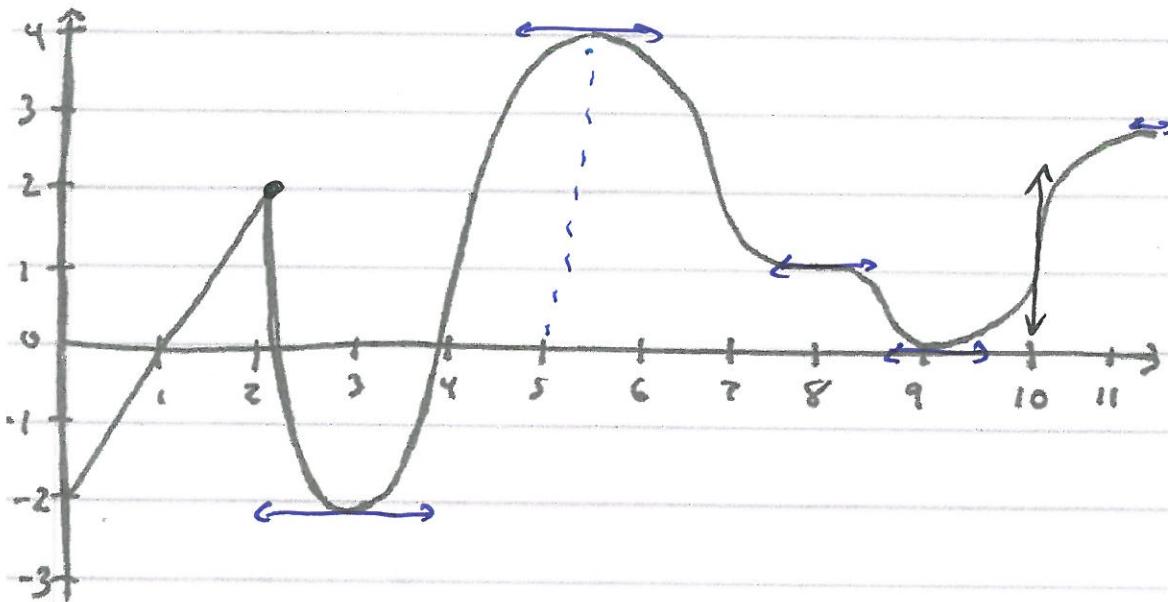
$$1 + 2x^{-\frac{2}{3}} = 0$$

$$2x^{-\frac{2}{3}} = -1$$

$$2 = -x^{\frac{2}{3}} \Rightarrow -2 = x^{\frac{2}{3}} \Rightarrow -8 = (-2)^3 = (x^{\frac{2}{3}})^3 = x^2$$

no real solutions

Example 4.2.5. Use the graph below to answer the following questions.



- a. Where is the derivative zero?

$$X = 3, 5, 8, 9, 11$$

- b. Where is the derivative undefined?

$$X = 2, 10$$

- c. What are the critical points?

$$X = 2, 3, 5, 8, 9, 10, 11$$

- d. What is the global maximum?

$$4$$

- e. What is the global minimum?

$$-2$$

What points $x = c$ could give local minimums or local maximums?

$$f'(c) = 0 \text{ or undefined or endpoints}$$

What points $x = c$ could give extreme values?

$$\text{Endpoints, } f'(c) = 0 \text{ or undefined}$$

Note: Some CPs may not correspond to local extrema

Theorem 4.2.1 (Extreme Value Theorem (EVT)). A continuous function on a closed interval has a global maximum and global minimum. Further, these occur at

endpoints or CPs

Example 4.2.6. What are the extreme values of $f(x) = xe^{2x}$ on $[-2, 2]$.

$$\text{CP for } f(x): x = -\frac{1}{2}$$

$$f(-2) = -2e^{-4} \approx -0.037$$

$$f\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-1} \approx -0.184 \rightarrow \text{minimum}$$

$$f(2) = 2e^4 \rightarrow \text{maximum}$$

Example 4.2.7. What are the extreme values of $g(x) = |x^2 + 4x - 12|$ on $[-5, 3]$.

$$\text{CP for } g(x): x = -6, -2, 2$$

$$g(-5) = 7$$

$$g(-2) = +16 \rightarrow \text{local max}$$

$$g(2) = 0 \rightarrow \text{min} \quad g(3) = 9 \quad \cancel{\text{max}}$$

Example 4.2.8. What are the extreme values of $k(x) = 4x - \sqrt{x^2 + 1}$ on $[0, 7]$.

$$k'(x) = 4 - \frac{1}{2} (x^2 + 1)^{-1/2} (2x)$$

$$= 4 - \frac{x}{\sqrt{x^2 + 1}}$$

$$0 = 4 - \frac{x}{\sqrt{x^2 + 1}}$$

$$0 = 4\sqrt{x^2 + 1} - x$$

$$x = 4\sqrt{x^2 + 1}$$

$$x^2 = 4(4(x^2 + 1)) = 16x^2 + 16$$

$$-16 = 15x^2$$

No real solutions

$$k(0) = -1 \leftarrow \min$$

$$k(7) = 28 - \sqrt{50} > 0$$

\uparrow
max

Example 4.2.9. What are the extreme values of $l(x) = \frac{1-x}{x^2+3x}$ on $[1, 4]$.

Check domain: $x^2+3x=0 \rightarrow x(x+3)$
 $x=0, -3$

$$\begin{aligned} f'(x) &= \frac{-(x^2+3x)-(2x+3)(1-x)}{(x^2+3x)^2} \\ &= \frac{-x^2-3x-(2x-2x^2+3-3x)}{(x^2+3x)^2} \\ &= \frac{x^2-2x-3}{(x^2+3x)^2} = \frac{(x-3)(x+1)}{(x^2+3x)^2} \end{aligned}$$

$$f'(x)=0 \iff x=3, -1$$

$$f(1)=0 \leftarrow \text{max}$$

$$f(3)=-\frac{1}{9} \leftarrow \text{min}$$

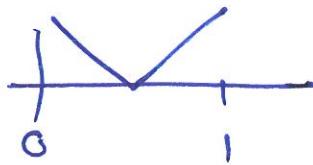
$$f(4)=-\frac{3}{28}$$

$$f(x)=l(x)$$

True or False: If $f(x)$ is continuous on a closed interval, then it is enough to look at the points where $f'(x) = 0$ in order to find its global maximum and minimum.

check endpoints

$f'(x)$ undefined



Let $f(x)$ be a differentiable function on a closed interval with $x = a$ being one of the endpoints of the interval. If $f'(a) > 0$, then

(a) f could have either a global max or min at $x = a$ \times

(b) f cannot have a global max at $x = a$

(c) f cannot have a global min at $x = a$



$$f(a) < f(x)$$

