3.9 General Exponential and Logarithmic Functions

Theorem 3.9.1 (General Exponential and Logarithmic Derivatives).

$$\frac{d}{dx}b^x = b^x(\ln b) \qquad b>0$$

$$\frac{d}{dx}e^x = e^x(\ln e) = e^x$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}$$

Logarithmic Differentiation:

$$f'(x) = f(x)\cdot\frac{d}{dx}(\ln(f(x)))$$

$$\frac{d}{dx}x^x \quad (f(x)=x^x)$$

$$f'(x) = f(x)(\ln f(x))' = x^x(\ln x^x)' = x^x(x\ln x)'$$

$$= x^x\left(\ln x + \left(\tfrac{1}{x}\right)x\right) = x^x(\ln x + 1)$$

Example 3.9.1. $y = t\ln(t) - t$. What is y'?

$$y' = (1)(\ln(t)) + \frac{1}{t}\cdot t - 1$$

$$= \ln t + 1 - 1$$

$$= \ln t$$

Example 3.9.2. Find the derivative of 7^{4x-x^2}.

$$f(x) = 7^{4x-x^2}$$

$$f'(x) = (\ln 7)\, 7^{4x-x^2}\,(4-2x)$$

$$= (4-2x)(\ln 7)\, 7^{4x-x^2}$$

$$\boxed{\frac{d}{dx} b^{f(x)} = f'(x)\,(\ln b)\, b^{f(x)}}$$

Example 3.9.3. Find the derivative of $3^x \ln(1+x^2)$.

$$f(x) = 3^x \ln(1+x^2)$$

$$f'(x) = (3^x \ln 3)\ln(1+x^2) + \frac{2x}{1+x^2}\cdot 3^x$$

$$= \ln 3 \ln(1+x^2)\, 3^x + \frac{2x3^x}{1+x^2}$$

Example 3.9.4. Find the derivative of $(x+1)(x+2)^2(x+3)^3(x+4)^4(x+5)^5$.

$$f(x) = (x+1)(x+2)^2(x+3)^3(x+4)^4(x+5)^5$$

$$f'(x) = f(x)\left(\ln f(x)\right)'$$

$$= f(x)\left(\ln(x+1) + 2\ln(x+2) + 3\ln(x+3) + 4\ln(x+4) + 5\ln(x+5)\right)'$$

$$= f(x)\left(\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} + \frac{4}{x+4} + \frac{5}{x+5}\right)$$

$$= (x+2)^2(x+3)^3(x+4)^4(x+5)^5$$

$$+ 2(x+1)(x+2)(x+3)^3(x+4)^4(x+5)^5$$

$$+ 3(x+1)(x+2)^2(x+3)^2(x+4)^4(x+5)^5$$

$$+ 4(x+1)(x+2)^2(x+3)^3(x+4)^3(x+5)^5$$

$$+ 5(x+1)(x+2)^2(x+3)^3(x+4)^4(x+5)^4$$

$$\frac{(\cancel{x+1})(x+2)^2(x+3)^3(x+4)^4(x+5)^5}{\cancel{x+1}}$$

Example 3.9.5. Find the derivative of $x^{\cos x}$.

$$f(x) = x^{\cos x}$$

$$\frac{\cos x \, x^{\cos x}}{x}$$

$$f'(x) = f(x)\left(\ln f(x)\right)'$$

$$= x^{\cos x}\left(\ln x^{\cos x}\right)'$$

$$= x^{\cos x}\left(\cos x \ln x\right)'$$

$$= x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

$$= -\sin x \, x^{\cos x} \ln x + \cos x \, x^{\cos x - 1}$$

$$\frac{d}{dx} e^{\cos x} = -\sin x\, e^{\cos x}$$

$$\frac{d}{dx} x^{\cos x} \neq -\sin x\, x^{\cos x} \ln x$$

$$\frac{d}{dx} x^{\cos x} \neq \cos x\, x^{\cos x - 1}$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$