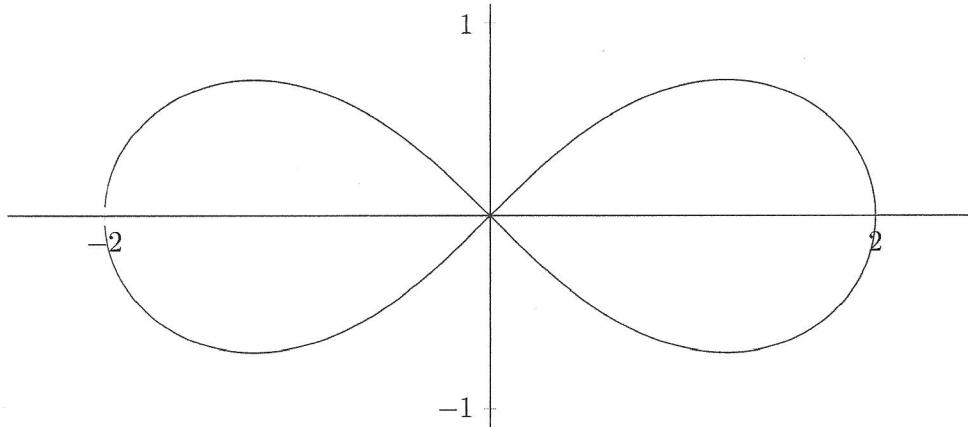


3.8 Implicit Differentiation

Definition 3.8.1. If we can write y as a function of x (i.e. $y = f(x)$), then we say y is given *explicitly*. If we cannot, we say that y is given *implicitly*.

Example 3.8.1. $(x^2 + y^2)^2 = 4(x^2 - y^2)$



Example 3.8.2. $y^4 - 2y = 4x^3 + x$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^4 - 2y) = 4y^3 \frac{dy}{dx} - 2 \frac{dy}{dx} = (4y^3 - 2) \frac{dy}{dx}$$

$$\frac{d}{dx}(4x^3 + x) = 12x^2 \frac{dx}{dx} + \frac{dx}{dx} = 12x^2 + 1$$

$$\frac{dy}{dx} = \frac{12x^2 + 1}{4y^3 - 2}$$

Example 3.8.3. Find $\frac{dy}{dx}$ when $\tan(x^2y) = (x+y)^3$.

$$\boxed{\frac{d}{dx} \tan(x^2y) = \sec^2(x^2y) \left(2xy + \frac{dy}{dx} x^2 \right)}$$

$$\boxed{\frac{d}{dx} (x+y)^3 = 3(x+y)^2 \left(1 + \frac{dy}{dx} \right)}$$

$$2xy\sec^2(x^2y) + x^2\sec^2(x^2y)\frac{dy}{dx} = 3(x+y)^2 + 3(x+y)^2 \frac{dy}{dx}$$

$$x^2\sec^2(x^2y)\frac{dy}{dx} - 3(x+y)^2 \frac{dy}{dx} = 3(x+y)^2 - 2xy\sec^2(x^2y)$$

$$\boxed{\frac{dy}{dx} = \frac{3(x+y)^2 - 2xy\sec^2(x^2y)}{x^2\sec^2(x^2y) - 3(x+y)^2}}$$

Example 3.8.4. Find the equation of the tangent line to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at $(1, 1)$.

$$y = f'(a)(x-a) + f(a)$$

$$\frac{d}{dx} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}\frac{dy}{dx}y^{-\frac{1}{3}} = \frac{d}{dx} 2 = 0$$

$$\text{At } (1, 1): \quad \frac{2}{3} + \frac{2}{3} \frac{dy}{dx} \Big|_{(1,1)} = 0 \quad \Rightarrow \quad \frac{2}{3} \frac{dy}{dx} = -\frac{2}{3}$$

$$\frac{dy}{dx} \Big|_{(1,1)} = -1$$

$$y - 1 = -(x-1) = -x + 1$$

$$y = -x + 2$$

Example 3.8.5. $(x^2 + y^2)^2 = 4(x^2 - y^2)$. Find coordinates of points where the tangent line is horizontal.

$$\frac{d}{dx} (x^2 + y^2)^2 = 2(x^2 + y^2)(2x + 2yy')$$

$$\frac{d}{dx} 4(x^2 - y^2) = 4(2x - 2yy')$$

$$\begin{cases} x \neq 0 \\ x^2 + y^2 = 1 \end{cases}$$

$$\text{Want } y' = 0$$

$$\Rightarrow 2(x^2 + y^2)(2x) = 4(2x)$$

$$(x^2 + y^2) = 2 \quad (x \neq 0) \quad x = 0 \quad \text{any } y \text{ works}$$

$$y = \pm \sqrt{2 - x^2}$$

$$(x^2 + (2 - x^2))^2 = 4(x^2 - (2 - x^2)) = 4(2x^2 - 2)$$

$$1 = 2x^2 - 2 \Rightarrow 2x^2 = 3 \Rightarrow x^2 = \frac{3}{2} \Rightarrow x = \pm \sqrt{\frac{3}{2}}$$

$$y^4 = \pm 4y^2 \Rightarrow \text{only imaginary solns}$$

$$y = \pm \sqrt{2 - \frac{3}{2}} = \pm \sqrt{\frac{1}{2}}$$

Example 3.8.6. Which of the following are incorrect?

1. $\frac{d}{dx} x^2 y = 2xy + x^2 y'$
2. $\frac{d}{dx} xe^y = e^y + xe^y y'$
3. $\frac{d}{dx} (x+y)^5 = 1 + 5y^4$
 $= 5(x+y)^4(1+y')$

$$\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}} \right), \left(\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}} \right),$$

$$\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{1}{2}} \right), \left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}} \right)$$

Example 3.8.7. Compute y'' at the point $(1, 1)$ on $x^3 + y^3 = 3x + y - 2$.

$$\frac{d}{dx}(x^3 + y^3) = 3x^2 + 3y^2 y'$$

$$\frac{d}{dx}(3x + y - 2) = 3 + y'$$

$$(6y y') y' + 3y^2 y''$$

$$\frac{d}{dx}(3y^2 y') = (6y^2) y' + (y'')(3y^2)$$

$$\frac{d}{dx}(3x^2 + 3y^2 y') = 6x + 6y(y')^2 + y''(3y^2)$$

$$\frac{d}{dx}(3 + y) = y''$$

$$\Rightarrow 2y' = 0$$

$$3 + y' = 3 + 3y' \Rightarrow y' = 3y' \Rightarrow y' = 0$$

$$6 + y''(3) = y'' \Rightarrow 2y'' = -6$$

$$\Rightarrow y'' = -3$$

Theorem 3.8.1 (Derivatives of Inverse Trig Functions).

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \cot^{-1}(x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Example 3.8.8. $\frac{d}{dx} e^{\cos^{-1}(x)}$

$$= e^{\cos^{-1}(x)} (\cos^{-1}(x))'$$

$$= e^{\cos^{-1}(x)} \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$= \frac{-e^{\cos^{-1}(x)}}{\sqrt{1-x^2}}$$

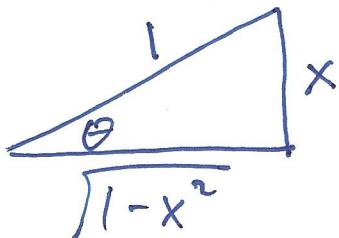
Example 3.8.9. $\frac{d}{dt} \sec^{-1}(5t^2)$

$$= \frac{10t}{|5t^2| \sqrt{(5t^2)^2 - 1}}$$

$$= \frac{10t}{5t^2 \sqrt{25t^4 - 1}}$$

$$\Rightarrow \frac{2}{t \sqrt{25t^4 - 1}}$$

Example 3.8.10. Why is $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$?



$$\sin^{-1}(\sin \theta) = \theta \quad f(y) = \cancel{\sin^{-1}} \sin^{-1}(y)$$

$$\frac{d}{d\theta} \sin^{-1}(\sin \theta) = \cancel{f'(y)} f'(\sin \theta) \cos \theta$$

$$\frac{d}{d\theta} \theta = 1$$

$$\Rightarrow f'(\sin \theta) = \frac{1}{\cos \theta} = \frac{1}{\frac{\sqrt{1-x^2}}{1}} = \frac{1}{\sqrt{1-x^2}}$$

$$f'(x)$$